

Evolution of cosmological perturbations and the production of non-Gaussianities through a nonsingular bounce: Indications for a no-go theorem in single field matter bounce cosmologies

Jerome Quintin,^{1,*} Zeinab Sherkatghanad,^{1,2,†} Yi-Fu Cai,^{3,‡} and Robert H. Brandenberger^{1,§}

¹*Department of Physics, McGill University, Montréal, Québec H3A 2T8, Canada*

²*Department of Physics, Isfahan University of Technology, Isfahan, 84156-83111, Iran*

³*CAS Key Laboratory for Researches in Galaxies and Cosmology,
Department of Astronomy, University of Science and Technology of China,
Chinese Academy of Sciences, Hefei, Anhui 230026, China*

Assuming that curvature perturbations and gravitational waves originally arise from vacuum fluctuations in a matter-dominated phase of contraction, we study the dynamics of the cosmological perturbations evolving through a nonsingular bouncing phase described by a generic single scalar field Lagrangian minimally coupled to Einstein gravity. In order for such a model to be consistent with the current upper limits on the tensor-to-scalar ratio, there must be an enhancement of the curvature fluctuations during the bounce phase. We show that, while it remains possible to enlarge the amplitude of curvature perturbations due to the nontrivial background evolution, this growth is very limited because of the conservation of curvature perturbations on super-Hubble scales. We further perform a general analysis of the evolution of primordial non-Gaussianities through the bounce phase. By studying the general form of the bispectrum we show that the non-Gaussianity parameter f_{NL} (which is of order unity before the bounce phase) is enhanced during the bounce phase if the curvature fluctuations grow. Hence, in such nonsingular bounce models with matter given by a single scalar field, there appears to be a tension between obtaining a small enough tensor-to-scalar ratio and not obtaining a value of f_{NL} in excess of the current upper bounds. *This conclusion may be considered as a “no-go” theorem that rules out any single field matter bounce cosmology starting with vacuum initial conditions for the fluctuations.*

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*Electronic address: jquintin@physics.mcgill.ca

†Electronic address: z.sherkat@mail.mcgill.ca

‡Electronic address: yifucai@ustc.edu.cn

§Electronic address: rhb@hep.physics.mcgill.ca

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I. INTRODUCTION

As was realized in [1, 2], there is a duality between the evolution of curvature fluctuations in an exponentially expanding universe and in a contracting universe with the equation of state of matter. In both cases, curvature fluctuations which originate as quantum vacuum perturbations on sub-Hubble scales acquire a scale-invariant spectrum at later times on super-Hubble scales. The observed small red tilt of the spectrum of curvature perturbations which has now been confirmed by observations (see e.g. [3, 4]) can be obtained in an expanding universe by a slow decrease of the Hubble constant during the period of inflation [5], whereas in a matter-dominated phase of contraction a small cosmological constant (with magnitude comparable to what is needed to explain today’s dark energy) yields the same tilt [6] (see alternatively [7]). To avoid reaching a singularity at the end of the contracting phase, it is necessary to either modify gravity or consider matter violating the null energy condition (NEC). Then it is possible to obtain nonsingular bouncing cosmologies which have the potential to yield an explanation for the structures in the universe which we now observe. This scenario of structure formation alternative to inflation is called the “matter bounce” scenario (see e.g. [8, 9] for reviews).

Examples of modified gravity models which yield bouncing cosmologies include the “nonsingular Universe” construction of [10, 11], nonlocal gravity actions like [12], or Hořava-Lifshitz gravity [13]. It is in general very hard to study the evolution of fluctuations in these models. We will hence focus on models in which the bounce is obtained from the matter sector. One method of obtaining a nonsingular bounce with a single scalar field involves the formation of a ghost condensate during the bounce phase (see [14–19] for initial developments). A general problem for bouncing cosmologies is the Belinsky-Khalatnikov-Lifshitz (BKL) instability [20], the fact that the energy density in the form of anisotropies will explode and destroy the homogeneous bounce [21]. This problem can be “solved” by endowing the scalar matter field with a negative potential which leads to an Ekpyrotic phase of contraction before the bounce [22–24] and hence can mitigate the anisotropy problem [25]¹.

In the matter bounce scenario, primordial quantum fluctuations exit the Hubble horizon while the universe is in a matter-dominated contracting phase and the resulting power spectrum of curvature perturbations is scale-invariant [1, 2]. On the other hand, the gravitational wave mode obeys the same equation of motion on super-Hubble scales as the curvature perturbations (considering the canonical variables in each case). Hence, before the bounce phase the tensor-to-scalar ratio r would be of order unity. Thus, if the perturbations passed through the nonsingular bounce unchanged, it would imply that curvature perturbations and primordial gravitational waves would have the same amplitude after the bounce. In terms of the tensor-to-scalar ratio, it would mean that $r \sim \mathcal{O}(1)$, well above current observational bounds [3, 4, 27].

¹ Such a negative potential may arise from the standard model Higgs field since, based on the recent Higgs and top quark mass measurements, the standard model Higgs develops an instability at large field values (in the absence of new physics) [26].

Since the curvature fluctuations couple nontrivially to matter during the bounce phase, whereas the tensor perturbations are determined simply by the evolution of the scale factor $a(t)$, one may expect that the curvature perturbations would be enhanced relative to the tensor modes during the bounce. In fact, early calculations indicated that curvature perturbations grew exponentially during the bounce phase, hence suppressing the tensor-to-scalar ratio [22, 28]. A more recent study [29] numerically explored the evolution of scalar fluctuations through a nonsingular bounce model similar to the one studied in [22] and found no enhancement of curvature perturbations through the bounce. In light of the relevance of a possible enhancement of the curvature fluctuations for the predicted value of the tensor-to-scalar ratio, the growth of curvature fluctuations during a nonsingular bounce needs to be reconsidered. This is what we aim to do in this paper.

The second goal of this paper is to carefully track the evolution of the three-point function (bispectrum) of curvature perturbations through the bounce. In earlier work [30] it was shown that the bispectrum of curvature fluctuations before the bounce phase has an amplitude of the order $f_{\text{NL}} \sim \mathcal{O}(1)$ with a specific shape. As we argued above, if the perturbations were to pass through the nonsingular bounce unchanged, it would imply a large tensor-to-scalar ratio in excess of the observational bounds. On the other hand, if curvature perturbations were to experience a nontrivial growth through the bounce, one should expect additional nonzero contributions to the bispectrum coming from the bounce phase, and there would then be the danger that the final amplitude of the bispectrum exceeds the observational upper bounds from [31, 32]. Thus, a potential conflict looms: either the tensor-to-scalar ratio is too large, or else the non-Gaussianities exceed observational bounds. This problem has indeed already been found in a model of a nonsingular bouncing cosmology in which a nonvanishing positive spatial curvature is responsible for the bounce [33, 34]. We will study this issue in the context of the more realistic models in which the nonsingular bounce is generated by the matter sector. In particular, we will explore the question in the context of a ghost-condensate bounce.

We will indeed demonstrate that, at least in our model, the evolution of the curvature perturbations in the bounce phase connects the value of the tensor-to-scalar ratio with the amplitude of non-Gaussianities. The suppression of the tensor-to-scalar ratio to restore compatibility with the observational bounds requires an enhancement of the curvature fluctuations during the bounce phase. Such an enhancement will increase the magnitude of the non-Gaussianities to a level inconsistent with the observational bounds on the amplitude of the bispectrum. Based on our result we conjecture that there exists a “no-go” theorem in single field nonsingular matter bounce cosmologies which relates the tensor-to-scalar ratio and non-Gaussianities, preventing these models to satisfy the current observational bounds. A tensor-to-scalar ratio below current observational bounds would imply a too large amplitude of non-Gaussianities, whereas non-Gaussianities of order $f_{\text{NL}} \sim \mathcal{O}(1)$ would imply a too large amplitude of the primordial gravitational wave spectrum. Therefore, a single field nonsingular matter bounce cannot be made consistent with current observations if the primordial perturbations arise from vacuum initial conditions.

Our analysis assumes that both curvature perturbations and gravitational waves originate as quantum vacuum fluctuations in the initial phase of contraction. A model with thermal fluctuations (as obtained for example in the context of string gas cosmology [35, 36]) will easily avoid our “no-go” theorem. As shown in [37–40], we obtain a tensor-to-scalar ratio much smaller than order unity while obtaining non-Gaussianities which are negligible on cosmological scales [41].

The paper is organized as follows. We first start with a short review of cosmological perturbation theory in Sec. II. We then motivate the idea of the no-go theorem proposed in this paper in Sec. III. In Sec. IV, we briefly review the general picture of bouncing cosmology in terms of a single scalar field of Galileon type. After that, in Sec. V we analyze the perturbation equation for primordial curvature perturbations at linear order during the nonsingular bouncing phase. We point out under which conditions there can be an enhancement of their amplitude. Then in Sec. VI, we perform a detailed analysis of the bispectrum generated in the bouncing phase of our specific model. We combine the analyses of scalar and tensor perturbations together with non-Gaussianities in Sec. VII, and we show how current observational bounds severely constrain the parameter space of the single field bouncing model. The analysis is expected to hold quite generally for single field matter bounce cosmologies. We conclude with a discussion in Sec. VIII. Throughout this paper, we adopt the mostly minus convention for the metric and define the reduced Planck mass as $M_p^2 \equiv 1/8\pi G_N$ where G_N is Newton’s gravitational constant.

II. A BRIEF REVIEW OF COSMOLOGICAL PERTURBATION THEORY

Linear perturbations of the metric about a homogeneous and isotropic background space-time can be decomposed into scalar, vector, and tensor modes (see [42] for a review of the theory of cosmological perturbations and [43] for an introductory overview). The scalar modes are those which couple to matter energy density and pressure perturbations. We call these the *cosmological perturbations*. Tensor modes exist in the absence of matter - they correspond to gravitational waves. In the case of matter without anisotropic stress at linear order in the amplitude of the fluctuations, there is only one physical degree of freedom for the scalar fluctuations. For the purpose of computations it is often

convenient to work in the conformal Newtonian gauge (coordinate system) in which the perturbed metric for scalar modes reads

$$ds^2 = a^2(\eta) ([1 + 2\Phi(\eta, \vec{x})] d\eta^2 - [1 - 2\Phi(\eta, \vec{x})] d\vec{x}^2) , \quad (1)$$

where η denotes conformal time, $a(\eta)$ is the cosmological scale factor, \vec{x} represents comoving spatial coordinates, and Φ denotes the gravitational potential. For tensor modes, the perturbed metric reads

$$ds^2 = a^2(\eta) (d\eta^2 - [\delta_{ij} + h_{ij}(\eta, \vec{x})] dx^i dx^j) , \quad (2)$$

where h_{ij} is trace-free and divergenceless.

Let us consider the matter content to be described by a single scalar field of canonical form with Lagrangian density

$$\mathcal{L}_m = \frac{1}{2} M_p^2 g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) . \quad (3)$$

Note that we take the scalar field to be dimensionless throughout this paper as a convention. Linear perturbations of the scalar field then have the form

$$\phi(\eta, \vec{x}) = \phi_0(\eta) + \delta\phi(\eta, \vec{x}) , \quad (4)$$

where ϕ_0 is the unperturbed homogeneous part of ϕ . In the scalar sector, metric and matter perturbations couple to one another, so it is useful to define a linear combination of these perturbations,

$$\mathcal{R} \equiv \frac{\mathcal{H}}{\phi'_0} \delta\phi + \Phi . \quad (5)$$

There are two reasons for focusing on this variable. First of all, it gives the curvature fluctuation in comoving coordinates (coordinates in which the matter field is uniform), and is hence the variable we are interested in computing. Second, it is simply related to the Sasaki-Mukhanov [44, 45] variable v in terms of which the action for cosmological perturbations has canonical form. Note that in the above, $\mathcal{H} \equiv a'/a$ is the conformal Hubble parameter and a prime denotes a derivative with respect to conformal time. In fact, the Sasaki-Mukhanov variable is

$$v \equiv z\mathcal{R} , \quad (6)$$

with

$$z = a \frac{\phi'_0}{\mathcal{H}} M_p . \quad (7)$$

The equation of motion that results from expanding the perturbed action for gravity and matter to second order is given by

$$v_k'' + \left(c_s^2 k^2 - \frac{z''}{z} \right) v_k = 0 . \quad (8)$$

The equation is written in Fourier space, where k represents the comoving wave number of the curvature perturbations, and c_s is the speed of sound which is equal to one for a scalar field with canonical action (3). Similarly, for tensor modes the Mukhanov variable is

$$\mu \equiv ah , \quad (9)$$

where h is the amplitude of the polarization tensor h_{ij} (the two polarization states evolve independently at linear order and obey the same equation of motion) and the resulting equation of motion is

$$\mu_k'' + \left(c_s^2 k^2 - \frac{a''}{a} \right) \mu_k = 0 . \quad (10)$$

Alternatively, without the use of the Mukhanov variables, the equation of motion for curvature and tensor perturbations can be written as

$$\mathcal{R}_k'' + 2 \frac{z'}{z} \mathcal{R}_k' + c_s^2 k^2 \mathcal{R}_k = 0 , \quad (11)$$

$$h_k'' + 2 \frac{a'}{a} h_k' + c_s^2 k^2 h_k = 0 , \quad (12)$$

respectively.

Finally, let us introduce the scalar perturbation variable

$$\zeta \equiv \Phi + \frac{2}{3} \frac{\Phi' + \mathcal{H}\Phi}{\mathcal{H}(1+w)} , \quad (13)$$

where $w \equiv P/\rho$ is the equation of state parameter (P is the pressure and ρ is the energy density). On super-Hubble scales, i.e. for $k \ll \mathcal{H}$, this variable is equivalent to the curvature perturbation variable \mathcal{R}_k [46]. In other words, $\mathcal{R}_k = \zeta_k$, and thus, throughout the rest of this paper, we will use \mathcal{R}_k and ζ_k interchangeably to denote curvature perturbations on super-Hubble scales.

III. OUTLINE OF THE NO-GO CONJECTURE

As explained in the introduction, a careful study of the evolution of curvature perturbations and the production of non-Gaussianities during a nonsingular bounce may lead to a “no-go” theorem, the impossibility of obtaining a sufficiently small tensor-to-scalar ratio while maintaining a bispectrum with an amplitude smaller than the current observational bounds. In this section we will provide a qualitative analysis of this problem by giving simple estimates of the tensor-to-scalar ratio and of the amplitude of the bispectrum assuming that the curvature fluctuations undergo some growth through the bounce phase. We first start by setting up the matter bounce formalism.

A. Fluctuations in the matter bounce

In the matter bounce, primordial quantum fluctuations originate on sub-Hubble scales during a matter-dominated contracting phase and exit the Hubble radius during this phase. The perturbations then remain on super-Hubble scales as the universe contracts and passes through the bounce phase, except for a very small time interval right at the bounce point (at which time the Hubble radius goes to infinity). The fluctuations with wavelength of cosmological interest today will then reenter the Hubble radius in the standard radiation or matter-dominated expanding phases. If the bounce is completely symmetric, then fluctuations which exit the Hubble radius in the matter phase of contraction reenter the Hubble radius in the matter phase of expansion. However, we expect the bounce to be asymmetric and entropy to be generated during the bounce. In this case, the radiation phase of expansion is longer than the radiation phase of contraction.

To understand the evolution of quantum fluctuations in a contracting universe, one needs to determine the form of the variable z and then solve Eq. (8). Using the Friedmann equations, the time derivative of the Hubble parameter is given by

$$\dot{H} = -\frac{\dot{\phi}_0^2}{2} , \quad (14)$$

where a dot denotes a derivative with respect to cosmic time, t , and the subscript 0 indicates that we are referring to the background field. Defining the parameter ϵ ,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} , \quad (15)$$

and using Eq. (14), one finds

$$z = a \frac{\dot{\phi}_0}{H} M_p = a \sqrt{2\epsilon} M_p . \quad (16)$$

It is straightforward to show from the Friedmann equations that

$$\epsilon = \frac{3}{2}(1+w) , \quad (17)$$

so for a matter-dominated contracting universe with $w = 0$, we have $\epsilon = 3/2$. As a consequence, $z = a\sqrt{3}M_p$ and

$$\frac{z''}{z} = \frac{a''}{a} , \quad (18)$$

and we conclude that the scalar and tensor fluctuations evolve in exactly the same way. This is not true in general since w can vary in time. For example, in the case of inflationary cosmology, we recognize ϵ as the slow-roll parameter and it is time-dependent.

In a matter-dominated contracting universe, the scale factor scales as $a \sim (-t)^{2/3} \sim \eta^2$, and since $c_s^2 = 1$ for a canonical scalar field, the equation for the Sasaki-Mukhanov variable is

$$v_k'' + \left(k^2 - \frac{2}{\eta^2}\right) v_k = 0. \quad (19)$$

On super-Hubble scales, the k^2 term is negligible, and so the solution reads

$$v_k(\eta) = c_1 \eta^2 + c_2 \eta^{-1}. \quad (20)$$

Using the fact that $v_k = z \zeta_k$, the first term yields $\zeta_k \sim \text{constant}$, but in a contracting universe, the second term is the dominant solution,

$$\zeta_k \sim \eta^{-3}, \quad (21)$$

which implies that curvature perturbations grow in a contracting universe. In fact, the growth rate is precisely the correct one to convert an initial vacuum spectrum into a scale-invariant one (see e.g. [9] for a review).

B. Bound from the tensor-to-scalar ratio

The tensor-to-scalar ratio is defined as

$$r \equiv \frac{\mathcal{P}_t(k_*)}{\mathcal{P}_\zeta(k_*)}, \quad (22)$$

where k_* is the pivot scale which is used to parametrize the power spectra for tensor and curvature perturbations. The individual power spectra are defined by [47]

$$\mathcal{P}_t(k) = 2\mathcal{P}_h(k) \equiv 2 \times 16\pi \frac{k^3}{2\pi^2} |h_k|^2 = 16\pi \frac{k^3}{\pi^2} \frac{|\mu_k|^2}{a^2}, \quad (23)$$

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2} |\zeta_k|^2 = \frac{k^3}{2\pi^2} \frac{|v_k|^2}{z^2}, \quad (24)$$

respectively. The factor of 2 in the first step of the first line comes from the two polarization states of gravitons and the factor of 16π is a convention reflecting the fact that it is $16\pi M_p h$ which yields the canonical action of a free scalar field in an expanding background [42].

As we found in the previous subsection, $z = a\sqrt{3}M_p$ for the matter bounce, so the scalar power spectrum becomes

$$\mathcal{P}_\zeta(k) = \frac{k^3}{6\pi^2} \frac{|v_k|^2}{a^2 M_p^2}, \quad (25)$$

and furthermore, the tensor-to-scalar ratio becomes

$$r = 96\pi \left| \frac{\mu_{k_*}}{v_{k_*}} \right|^2 M_p^2. \quad (26)$$

where the factor M_p^2 reflects the fact that we have defined v_k to have dimensions of mass, whereas μ_k is dimensionless.

Since $z''/z = a''/a$ for the matter bounce, the evolution of scalar and tensor modes given by Eqs. (8) and (10), respectively, will be identical. In addition, if they originate from the same quantum vacuum, then $v_k(\eta) = M_p \mu_k(\eta)$. Consequently, we find that $r = 96\pi$. If perturbations passed through the bounce unchanged, it would result in $r = 96\pi$ at the beginning of the standard big bang cosmology phase which is three orders of magnitude larger than the current observational upper bound.

To gain some intuition on the effect of passing through the bounce phase, let us assume that curvature perturbations are enhanced by an amount $\Delta\zeta_k$ through the bounce, i.e.

$$\zeta_k(\eta_{B+}) = \zeta_k(\eta_{B-}) + \Delta\zeta_k, \quad (27)$$

where $\eta_{B\pm}$ denote the conformal time before $(-)$ and after $(+)$ the bounce. Then, the tensor-to-scalar ratio measured after the bounce becomes

$$r(\eta_{B+}) = 96\pi \left| \frac{h_{k_*}(\eta_{B+})}{\zeta_{k_*}(\eta_{B-}) + \Delta\zeta_{k_*}} \right|^2 . \quad (28)$$

Assuming that tensor modes remain constant through the bounce, i.e. $h_k(\eta_{B-}) = h_k(\eta_{B+})$, one finds that

$$\left| 1 + \frac{\Delta\zeta_{k_*}}{\zeta_{k_*}(\eta_{B-})} \right|^2 = \frac{r(\eta_{B-})}{r(\eta_{B+})} . \quad (29)$$

Taking the value of the tensor-to-scalar ratio before the bounce to be what we found earlier, i.e. $r(\eta_{B-}) = 96\pi$, and demanding that the tensor-to-scalar ratio is sufficiently suppressed after the bounce so that it satisfies the observational bound $r(\eta_{B+}) < 0.12$ (95% CL from [27, 48]), we find that curvature perturbations must be sufficiently enhanced during the bounce phase so that

$$\left| 1 + \frac{\Delta\zeta_{k_*}}{\zeta_{k_*}(\eta_{B-})} \right| \gtrsim 50.1 , \quad (30)$$

or using the triangle inequality,

$$\left| \frac{\Delta\zeta_{k_*}}{\zeta_{k_*}(\eta_{B-})} \right| \gtrsim 49.1 . \quad (31)$$

C. Bound from the bispectrum

The primordial bispectrum, B_ζ , is defined in terms of the three-point function as

$$\langle \zeta(\vec{k}_1) \zeta(\vec{k}_2) \zeta(\vec{k}_3) \rangle \equiv (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_\zeta(k_1, k_2, k_3) , \quad (32)$$

which we can rewrite as

$$\langle \zeta(\vec{k}_1) \zeta(\vec{k}_2) \zeta(\vec{k}_3) \rangle = (2\pi)^7 \delta^{(3)} \left(\sum_i \vec{k}_i \right) \frac{\mathcal{P}_\zeta^2}{\prod_i k_i^3} \mathcal{A}(k_1, k_2, k_3) , \quad (33)$$

where $k_i = |\vec{k}_i|$ and where the index i runs from 1 to 3. The function $\mathcal{A}(k_1, k_2, k_3)$ is known as the shape function and its amplitude defines the nonlinear parameter f_{NL} via

$$f_{\text{NL}}(k_1, k_2, k_3) = \frac{10}{3} \frac{\mathcal{A}(k_1, k_2, k_3)}{\sum_i k_i^3} . \quad (34)$$

Of particular interest is the local form of non-Gaussianities for which one of the three modes exits the Hubble radius much earlier than the other two, i.e. $k_1 \ll k_2 = k_3$. For this case, one can write

$$\zeta(\vec{x}) = \zeta_g(\vec{x}) + \frac{3}{5} f_{\text{NL}}^{\text{local}} \zeta_g(\vec{x})^2 , \quad (35)$$

where ζ_g is the Gaussian part of ζ .

In order to compute f_{NL} , one must evaluate the three-point function. To leading order in the interaction coupling constant, the three-point function is related to the interaction Lagrangian, L_{int} , via [49]

$$\langle \zeta(t, \vec{k}_1) \zeta(t, \vec{k}_2) \zeta(t, \vec{k}_3) \rangle = i \int_{t_i}^t d\tilde{t} \langle [\zeta(t, \vec{k}_1) \zeta(t, \vec{k}_2) \zeta(t, \vec{k}_3), L_{\text{int}}(\tilde{t})] \rangle , \quad (36)$$

where the square brackets denote the commutator and where t_i denotes the initial time before which there is no non-Gaussianity. The interaction Lagrangian is obtained by evaluating the action up to third order in perturbation theory

$$L_{\text{int}}(t) = \int d^3\vec{x} \mathcal{L}_3(t, \vec{x}) , \quad (37)$$

and for a canonical scalar field, the Lagrangian density for ζ to cubic order is given by [49]

$$\frac{\mathcal{L}_3}{M_p^2} = \left(\epsilon^2 - \frac{\epsilon^3}{2} \right) a^3 \zeta \dot{\zeta}^2 + \epsilon^2 a \zeta (\partial \zeta)^2 - 2\epsilon^2 a^3 \dot{\zeta} (\partial \zeta) (\partial \chi) + \frac{\epsilon^3}{2} a^3 \zeta (\partial_i \partial_j \chi)^2 + f(\zeta) \frac{\delta \mathcal{L}_2}{\delta \zeta}, \quad (38)$$

$$f(\zeta) = \frac{1}{4(aH)^2} (\partial \zeta)^2 - \frac{1}{4(aH)^2} \partial^{-2} \partial_i \partial_j (\partial_i \zeta \partial_j \zeta) - \frac{1}{H} \zeta \dot{\zeta} - \frac{\epsilon}{2H} \partial_i \zeta \partial_i \chi + \frac{\epsilon}{2H} \partial^{-2} \partial_i \partial_j (\partial_i \chi \partial_j \zeta), \quad (39)$$

where ∂^{-2} is the inverse Laplacian and where we define $\chi \equiv \partial^{-2} \dot{\zeta}$. Also, the equation of motion for ζ coming from the second order perturbed Lagrangian density \mathcal{L}_2 is given by

$$\frac{\delta \mathcal{L}_2}{\delta \zeta} = \frac{\partial}{\partial t} (a z^2 \dot{\zeta}) - \frac{c_s^2 z^2}{a} \partial^2 \zeta. \quad (40)$$

As we saw in Sec. III A, curvature perturbations grow on super-Hubble scales during the matter-dominated contracting phase until the bounce phase. While on super-Hubble scales the spatial gradient terms are negligible, i.e. $\partial_i \zeta, \partial_i \chi \simeq 0$, the growth in ζ implies that the interaction Lagrangian is dominated by

$$\frac{\mathcal{L}_3}{M_p^2} \simeq \left(\epsilon^2 - \frac{\epsilon^3}{2} \right) a^3 \zeta \dot{\zeta}^2 - \frac{1}{H} \zeta \dot{\zeta} \frac{\partial}{\partial t} (a z^2 \dot{\zeta}). \quad (41)$$

As was first shown in [30], the production of non-Gaussianities on a comoving scale k is dominated by the period between when the scale crosses the Hubble radius in the phase of matter contraction until the onset of the bounce phase, and the resulting non-Gaussianities are of order $f_{\text{NL}} \sim \mathcal{O}(1)$. For example, for the local shape, the authors of [30] found $f_{\text{NL}}^{\text{local}} = -35/16$.

Following what was done in the previous subsection, let us now assume that curvature perturbations grow during the bounce phase. For simplicity, let us assume that they grow linearly in time with constant rate

$$\dot{\zeta} = \frac{\Delta \zeta}{\Delta t_B}, \quad (42)$$

where the duration of the bounce is given by $\Delta t_B \equiv t_{B+} - t_{B-}$. Then, in the limit $k \rightarrow 0$ on super-Hubble scales, the contribution to the three-point function coming from the bounce phase is schematically given by

$$\langle \zeta(t_{B+})^3 \rangle_{\text{bounce}} \sim \frac{\zeta(t_{B+})^3}{M_p} \left(\frac{\Delta \zeta}{\Delta t_B} \right)^2 \int_{t_{B-}}^{t_{B+}} dt a(t)^3 \left[\epsilon(t)^2 - \frac{\epsilon(t)^3}{2} \right] \left[\zeta(t_{B-}) + \frac{\Delta \zeta}{\Delta t_B} (t - t_{B-}) \right], \quad (43)$$

and one expects that the dominant contribution to f_{NL} that results from evaluating the three-point function would scale as

$$f_{\text{NL}} \sim \frac{(\Delta \zeta)^2}{\Delta t_B} M_p^2, \quad (44)$$

plus terms of order $\Delta \zeta^1$ which would be subdominant for a large amplification $\Delta \zeta$.

We already see that a growth in the curvature perturbations during the bounce, $\Delta \zeta$, would enhance f_{NL} . From the previous subsection, we expect $\Delta \zeta$ to have a lower bound to match current observational bounds on r , and thus, we expect to find a lower bound on the amount of non-Gaussianities that are produced during the bounce phase. However, we cannot determine whether this contribution will be significant to $f_{\text{NL}} \sim \mathcal{O}(1)$ and whether the resulting lower bound will exceed current observational bounds without going into the details of the calculation.

D. The no-go theorem

Now, let us state our conjecture.

Conjecture 1 *For quantum fluctuations originating from a matter-dominated contracting universe, an upper bound on the tensor-to-scalar ratio (r) is equivalent to a lower bound on the amplification of curvature perturbations ($\Delta \zeta / \zeta$) which in turn is equivalent to a lower bound on the amount of primordial non-Gaussianities (f_{NL}). Furthermore, if the initial quantum vacuum is a canonical Bunch-Davies vacuum with $c_s = 1$, if the nonsingular bounce phase is due to a single NEC violating scalar field, and if general relativity holds at all energy scales, then satisfying the current observational upper bound on the tensor-to-scalar ratio cannot be done without contradicting the current observational upper bounds on f_{NL} (and vice-versa).*

In the rest of this paper, we will give an example of realization of this conjecture.

IV. A BRIEF REVIEW OF SINGLE FIELD BOUNCING COSMOLOGY

In the context of Einstein gravity, matter which violates the null energy condition must be introduced in order to obtain a cosmological bounce. A simple toy model is quintom cosmology, i.e. a model in which a scalar field with opposite sign in the action compared to a usual scalar field is introduced, and it is arranged that this field comes to dominate late in the contracting phase, thus yielding a nonsingular bounce [50]. A specific realization of this can be obtained in the Lee-Wick theory [51]. These models, however, suffer from a ghost instability [52]. To avoid this instability (at least at the perturbative level) one can make use of the ghost condensation mechanism [16] or the Galileon construction [18, 19].² These mechanisms involve a modified kinetic term in the action.

As mentioned in the introduction, bouncing models typically also suffer from the anisotropy problem, and to mitigate this problem, one can build into the scenario an Ekpyrotic phase of contraction which occurs at some point after the matter phase of contraction. Specifically, one can use a single scalar field with a kinetic term designed to yield a nonsingular bounce, and a potential energy function with a negative potential over some range of field values which is designed to yield Ekpyrotic contraction [22]. In this approach, a second scalar field with canonical kinetic term and with quadratic potential can be used to represent the regular matter of the Universe [23]. In this paper we will not consider the role which this second scalar field may play (for some ideas see [58]) but only consider the field ϕ which generates the Ekpyrotic contraction and the nonsingular bounce.

Throughout this paper, we assume only Einstein gravity plus matter. Thus, the action is given by

$$S = \int d^4x \sqrt{-g} \left(-\frac{M_p^2}{2} R + \mathcal{L}_m \right) , \quad (45)$$

where g is the determinant of the metric, R is the Ricci scalar, and \mathcal{L}_m is the matter Lagrangian. We assume that the matter content is dominated by only one scalar field (ϕ) before reheating (the energy density of matter created via reheating becomes only important after the bounce phase – see [24]). Thus, for the dynamics of the matter-dominated contracting era and the bounce phase to be described by second order equations of motion, we consider a Lagrangian of the most general form [59]

$$\mathcal{L}_m = K(\phi, X) + G(\phi, X) \square \phi + \mathcal{L}_4 + \mathcal{L}_5 , \quad (46)$$

where the kinetic variable X is defined as

$$X \equiv \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi , \quad (47)$$

and where the d'Alembertian operator is defined as

$$\square \phi \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \phi . \quad (48)$$

We do not write down the explicit form that \mathcal{L}_4 and \mathcal{L}_5 can take here, but the key point is that they involve higher order derivatives. If we assume that the energy scale at which the bounce occurs is low enough so that higher order derivative terms in the Lagrangian are negligible, then we can assume that $\mathcal{L}_4, \mathcal{L}_5 \approx 0$.

For the bounce to be nonsingular, the above Lagrangian must violate the null energy condition (NEC) at high energies. To do so, we assume the first term of the Lagrangian to have the form

$$K(\phi, X) = M_p^2 [1 - g(\phi)] X + \beta X^2 - V(\phi) , \quad (49)$$

where β is some positive constant. We see from Eq. (49) that when $g(\phi) > 1$, the sign of the kinetic term is reversed and a ghost condensate which violates the NEC is formed [14–16, 18, 19]. For this reason, one typically chooses the function $g(\phi)$ to have the form

$$g(\phi) = \frac{2g_0}{e^{-\sqrt{2/p}\phi} + e^{b_g \sqrt{2/p}\phi}} , \quad (50)$$

where p and b_g are positive constants. As $\phi \rightarrow 0$ at the bounce point, $g(\phi) \rightarrow g_0$, and the constant g_0 is naturally chosen to be $g_0 > 1$ to allow the NEC violation. We can also see from the form of $g(\phi)$ above that as ϕ goes away from

² Alternative possibilities of alleviating this instability may be achieved by considering various modified gravity implementations such as models of extended $F(R)$ gravity [53, 54], modified Gauss-Bonnet gravity [55], and torsion gravity scenarios [56, 57].

0 and as the kinetic variable X becomes small outside the bounce phase, $g(\phi)$ rapidly goes to 0 and the Lagrangian recovers its canonical form.

The potential $V(\phi)$ can be chosen in order to obtain an Ekpyrotic phase of contraction. This can be done by means of a potential which is negative for small values of $|\phi|$, but which approaches $V = 0$ exponentially at large positive and negative field values. Specifically, we have chosen the potential

$$V(\phi) = -\frac{2V_0}{e^{-\sqrt{2/q}\phi} + e^{b_V\sqrt{2/q}\phi}}, \quad (51)$$

where V_0 , q , and b_V are positive constants. Without the second term in the denominator, one obtains the potential postulated in the Ekpyrotic scenario [60].

One can then parametrize the background evolution during the bounce phase as follows. The Hubble parameter grows linearly in time, passing through zero at the time $t = t_B$ (the bounce point),

$$H(t) = \Upsilon(t - t_B), \quad (52)$$

where Υ is a positive constant. The scale factor immediately follows,

$$a(t) = a_B e^{\Upsilon(t-t_B)^2/2}. \quad (53)$$

Also, the scalar field evolves as

$$\dot{\phi}(t) = \dot{\phi}_B e^{-(t-t_B)^2/T^2}. \quad (54)$$

Since a_B and t_B can be arbitrarily redefined, we see that the parameters which describe the bounce phase are Υ , $\dot{\phi}_B$, t_{B-} (or t_{B+} assuming a symmetric bounce), and T . First, Υ gives the growth rate of the Hubble parameter. Second, $\dot{\phi}_B$ gives the maximal growth rate of the scalar field. Third, $\Delta t_B/T$ gives the dimensionless duration of the bounce. They can be related to the Lagrangian parameters via (see [22, 23])

$$\dot{\phi}_B \simeq \sqrt{\frac{2(g_0 - 1)}{3\beta}} M_p, \quad (55)$$

$$T \simeq \frac{H_{B+}}{\Upsilon} \sqrt{\frac{2}{\ln(\dot{\phi}_B^2/6H_{B+}^2)}}, \quad (56)$$

where $H_{B+} = \Upsilon(t_{B+} - t_B)$.

Given the model we have discussed and the bounce solution which we have given in parametric form, we will now follow the evolution of the curvature fluctuation variable ζ through the nonsingular bounce phase.

V. EVOLUTION OF CURVATURE PERTURBATIONS DURING THE BOUNCE

As we saw in Sec. II, the equation of motion for curvature perturbations [Eq. (11)] can be written as

$$\mathcal{R}_k'' + \frac{(z^2)'}{z^2} \mathcal{R}_k' + c_s^2 k^2 \mathcal{R}_k = 0. \quad (57)$$

For a noncanonical Lagrangian of the form of Eq. (46), the variable z and the sound speed are given by [22]

$$z^2 = \frac{2M_p^4 a^2 \dot{\phi}^2 \mathcal{P}}{(2M_p^2 H - G_{,X} \dot{\phi}^3)^2}, \quad (58)$$

$$c_s^2 = \frac{1}{\mathcal{P}} \left[K_{,X} + 4H \dot{\phi} G_{,X} - \frac{G_{,X}^2 \dot{\phi}^4}{2M_p^2} - 2G_{,\phi} + G_{,X\phi} \dot{\phi}^2 + (2G_{,X} + G_{,XX} \dot{\phi}^2) \ddot{\phi} \right], \quad (59)$$

where a comma denotes a partial derivative and where we defined

$$\mathcal{P} \equiv K_{,X} + \dot{\phi}^2 K_{,XX} + \frac{3}{2M_p^2} \dot{\phi}^4 G_{,X}^2 + 6H \dot{\phi} G_{,X} + 3H \dot{\phi}^3 G_{,XX} - 2G_{,\phi} - \dot{\phi}^2 G_{,\phi X}. \quad (60)$$

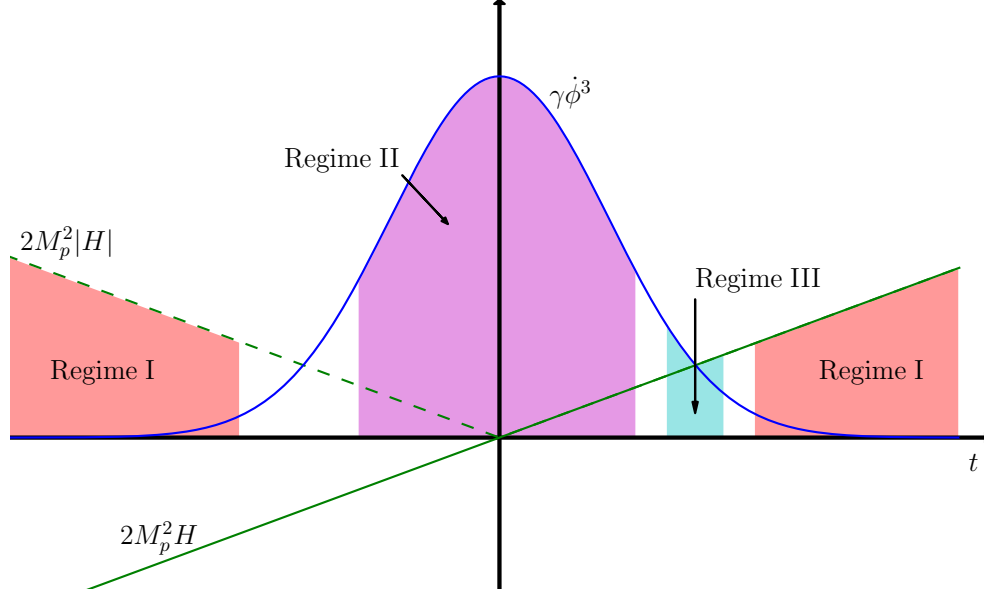


FIG. 1: Sketch of the different regimes in the bounce phase (not to scale). The horizontal axis represents physical time. The green solid curve shows $2M_p^2 H(t)$ and the dashed version depicts its absolute value. The bell-shaped blue curve represent $|G_{,X}(t)|\dot{\phi}^3(t)$, where we take $G_{,X} = \gamma$ to be a positive constant for simplicity. Regimes I, II, and III, defined by Eqs. (63), (64), and (65), are depicted by the pink, purple, and cyan regions, respectively.

As explained in Sec. III A, the perturbation modes that are of cosmological interest today were on super-Hubble scales during the bounce phase (except in the immediate vicinity of the bounce point), and thus we are most interested in the infrared (IR) regime of Eq. (57). In the limit $k \ll \mathcal{H}$, and recalling that \mathcal{R}_k and ζ_k are equivalent quantities in this limit, the equation that we want to solve is

$$\frac{d\zeta'}{d\eta} + \frac{(z^2)'}{z^2} \zeta' = 0, \quad (61)$$

where we drop the k index when it is clear that we are on super-Hubble scales. It is obvious from the above equation that one solution is the constant mode solution, $\zeta' = 0$, that one expects on super-Hubble scales, e.g. in inflation [61, 62] (see, however, [63]). More generally, the solution to Eq. (61) can be written as

$$\zeta'(\eta) = \zeta'(\eta_i) \frac{z^2(\eta_i)}{z^2(\eta)}, \quad (62)$$

where η_i denotes the initial time where the initial conditions are set. The evolution of ζ is thus governed by the evolution of z^2 , and we notice from the denominator of Eq. (58) that the evolution of z^2 has different regimes of interest:

$$\text{Regime I: } 2M_p^2 |H(t)| \gg |G_{,X}(t)|\dot{\phi}^3(t), \quad (63)$$

$$\text{Regime II: } 2M_p^2 |H(t)| \ll |G_{,X}(t)|\dot{\phi}^3(t), \quad (64)$$

$$\text{Regime III: } 2M_p^2 H(t) \approx G_{,X}(t)\dot{\phi}^3(t). \quad (65)$$

We represent these different regimes in Fig. 1, and we explore the consequences of each regime in the following subsections.

A. Evolution in Regime I

When Eq. (63) is valid, the expression for z^2 reduces to

$$z^2 \simeq \frac{M_p^2 a^2 \dot{\phi}^2}{H^2} \left(\frac{1 - g(\phi)}{2} \right). \quad (66)$$

Since the bounce phase is defined by $g(\phi) > 1$ and since z^2 must be positive to avoid ghost instabilities, the model parameters must be chosen such that this regime does not occur during the bounce phase. Outside the bouncing phase, the equation of motion in Regime 1 reduces to the standard one.

B. Evolution in Regime II

As the bounce point approaches, $H(t)$ goes to zero and we can expect Eq. (64) to be valid. To explore this regime, let us simplify the treatment by setting

$$G(\phi, X) = \gamma X \quad (67)$$

for some positive constant γ , so the regime becomes

$$2M_p^2 |H(t)| \ll \gamma \dot{\phi}^3(t). \quad (68)$$

Using the parametrizations introduced in the previous section, this condition can be rewritten as

$$|\Delta t| e^{3(\Delta t)^2/T^2} \ll \frac{\gamma \dot{\phi}_B^3}{2M_p^2 \Upsilon}, \quad (69)$$

where we defined $\Delta t \equiv t - t_B$. Since $\Delta t_B/T$ determines the dimensionless duration of the bounce, remaining close to the bounce is equivalent to demanding that $|\Delta t|/T \ll 1$. In particular, if we demand that

$$|\Delta t| \ll \min \left\{ \frac{T}{\sqrt{3}}, \frac{\gamma \dot{\phi}_B^3}{2M_p^2 \Upsilon} \right\}, \quad (70)$$

then it is ensured that we are in the regime set by Eq. (68). Thus, the expression for z^2 given in Eq. (58) reduces to

$$z^2(t) \simeq \frac{3\beta M_p^4 a^2(t)}{\gamma^2 \dot{\phi}^2(t)} \quad (71)$$

in this regime. In fact, there exists a time interval, which we define as $[t_{\text{amp-}}, t_{\text{amp+}}]$ with $t_{\text{amp}\pm} \equiv t_B \pm \Delta t_{\text{amp}}$, where the above approximation for $z^2(t)$ is certainly valid. We note that this expression is everywhere finite in that interval, so the solution to Eq. (61) can be directly written as

$$\dot{\zeta}(t) = \dot{\zeta}(t_i) \frac{a(t_i) z^2(t_i)}{a(t) z^2(t)}, \quad (72)$$

where the initial condition must be taken in the interval, i.e. $t_i \in [t_{\text{amp-}}, t_{\text{amp+}}]$, so logically we take $t_i = t_{\text{amp-}}$. Also, the solution will only be valid up to $t_{\text{amp+}}$. Inserting Eq. (71) and using the parametrizations introduced in the previous section, one finds

$$\begin{aligned} \zeta(t) &\simeq \zeta(t_{\text{amp-}}) + \dot{\zeta}(t_{\text{amp-}}) \int_{t_{\text{amp-}}}^t d\tilde{t} \left(\frac{a(t_{\text{amp-}})}{a(\tilde{t})} \right)^3 \left(\frac{\dot{\phi}(\tilde{t})}{\dot{\phi}(t_{\text{amp-}})} \right)^2 \\ &= \zeta(t_{\text{amp-}}) + \dot{\zeta}(t_{\text{amp-}}) \left(\frac{a(t_{\text{amp-}})}{a_B} \right)^3 \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\text{amp-}})} \right)^2 \int_{t_{\text{amp-}}}^t d\tilde{t} \exp \left[- \left(\frac{2}{T^2} + \frac{3}{2} \Upsilon \right) (\tilde{t} - t_B)^2 \right] \\ &= \zeta(t_{\text{amp-}}) + \dot{\zeta}(t_{\text{amp-}}) \left(\frac{a(t_{\text{amp-}})}{a_B} \right)^3 \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\text{amp-}})} \right)^2 T \sqrt{\frac{\pi}{8 + 6T^2 \Upsilon}} \\ &\quad \times \left[\text{erf} \left(\frac{t - t_B}{T} \sqrt{2 + \frac{3T^2 \Upsilon}{2}} \right) - \text{erf} \left(\frac{t_{\text{amp-}} - t_B}{T} \sqrt{2 + \frac{3T^2 \Upsilon}{2}} \right) \right]. \end{aligned} \quad (73)$$

Close to the bounce point, the scale factor remains nearly constant, so $a(t) \simeq a_B$. This implies that $\Upsilon(\Delta t)^2 \ll 2$, or in other words, that $H(t)\Delta t \ll \mathcal{O}(1)$. We will assume this to be valid throughout the rest of this paper whenever we are in the time interval $|\Delta t| \leq \Delta t_{\text{amp}}$. Therefore, the solution for $\zeta(t)$ reduces to

$$\zeta(t) \simeq \zeta(t_{\text{amp-}}) + \dot{\zeta}(t_{\text{amp-}}) \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\text{amp-}})} \right)^2 \frac{T\sqrt{2\pi}}{4} \left[\text{erf} \left(\frac{t - t_B}{T} \sqrt{2} \right) - \text{erf} \left(\frac{t_{\text{amp-}} - t_B}{T} \sqrt{2} \right) \right]. \quad (74)$$

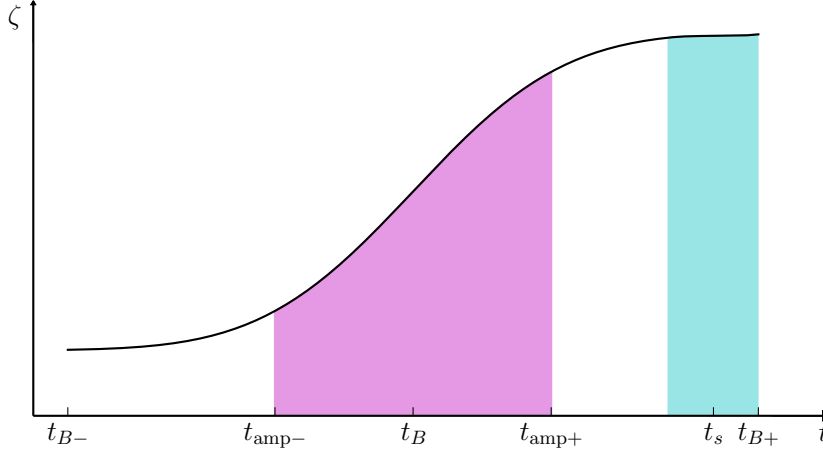


FIG. 2: Sketch of the evolution of curvature perturbation ζ on super-Hubble scales as a function of physical time t (not to scale). The beginning of the bounce phase, the bounce point, and the end of the bounce phase are denoted by t_{B-} , t_B , and t_{B+} , respectively. We defined $t_{\text{amp}\pm} \equiv t_B \pm \Delta t_{\text{amp}}$, and t_s is the time at which $z^2 \rightarrow \infty$. The purple region corresponds to Regime II of Fig. 1, where ζ grows at most linearly. The cyan region corresponds to Regime III of Fig. 1, where ζ is almost constant.

From the above solution, we see the constant mode and the growing mode. Whether the constant or the growing mode is dominant depends on many factors. For instance, the duration of this regime and the growth rate will play a crucial role. From the properties of the error function, we note that the growing mode grows at most linearly in time. Furthermore, the growth rate is maximal at the bounce point t_B and it is given by $\dot{\zeta}_{\text{max}} \simeq \dot{\zeta}(t_{\text{amp}-})[\dot{\phi}_B/\dot{\phi}(t_{\text{amp}-})]^2$.

C. Evolution in Regime III

One can notice from Eq. (62) that if $z^2 \rightarrow \infty$, then $\zeta' \rightarrow 0$, and curvature perturbations remain constant on super-Hubble scales. One can see from Eq. (58) that this happens at some physical time t_s (or η_s in conformal time) when

$$2M_p^2 H(t_s) = G_{,X}(t_s) \dot{\phi}^3(t_s). \quad (75)$$

At this point, the equation of motion for the curvature perturbations becomes singular, and furthermore, the Mukhanov variable $v_k = z\mathcal{R}_k$ diverges. For this reason, the evolution of the curvature perturbations has been explored in another gauge, the harmonic gauge (first introduced in the context of cosmological perturbation theory in [64]), where this singularity may disappear. Using the harmonic gauge, it has been shown in [29] that at η_s ,

$$\left. \frac{d\mathcal{R}_k}{d\eta} \right|_{\eta=\eta_s} = 0 \quad (76)$$

for all k modes. Carefully dealing with the singular equation of motion in the conformal Newtonian gauge, one can find that in the IR limit, the solution in conformal time close to the singular time η_s is (see Appendix A)

$$\zeta(\eta) = \zeta(\eta_i) + \zeta'(\eta_i) \left(\frac{(\eta - \eta_s)^3 + (\eta_s - \eta_i)^3}{3(\eta_s - \eta_i)^2} \right). \quad (77)$$

This indicates that perturbations can grow before the singular point (coming from Regime II), and that they could grow after the singular point (toward Regime I), but we saw that this regime is not present in the bounce phase (Sec. V A), so the bounce phase will end shortly after the singular point η_s .

D. Discussion

Let us summarize the evolution of curvature perturbations on super-Hubble scales through the bounce phase. Figure 2 is a sketch of the evolution of ζ according to the results found above. If ζ enters the bounce phase with

a nonvanishing time derivative³, then we find that curvature perturbations can grow at most linearly in some time interval $[t_{\text{amp}-}, t_{\text{amp}+}]$ and that the growth is maximal at the bounce point. This happens in what we call Regime II. We highlight this regime in purple in Fig. 2. Regime III follows Regime II at which point z^2 blows up and becomes infinite at some time t_s . At this point, curvature perturbations become constant, and the bounce phase ends shortly after. We highlight this regime in cyan in Fig. 2.

In the end, the amplification that ζ receives is dominated by the growth during the interval $[t_{\text{amp}-}, t_{\text{amp}+}]$. Between the beginning of the bounce phase and the beginning of the amplification phase, we expect little growth of the curvature perturbations, and so, the initial time derivative of ζ at the beginning of the amplification phase should be of the same order as the time derivative of ζ at the beginning of the bounce phase, which we expect to be small. In fact, in the Ekpyrotic phase of contraction (where $w \gg 1$) which precedes the bounce phase, the dominant mode of ζ is constant in time while the second mode is decaying (as shown in Appendix B). Hence, the amplitude of ζ at the end of the period of Ekpyrotic contraction is the same as the amplitude at the end of the matter phase of contraction (assuming for a moment that there is no intermediate radiation phase). Consequently, this could lead to a suppression of $\dot{\zeta}(t_{B-})$, and hence to a suppression of the growth of ζ in the bounce phase since, as we argued, $\dot{\zeta}(t_{B-}) \simeq \dot{\zeta}(t_{\text{amp}-})$.

The reason why we can match ζ and $\dot{\zeta}$ at the end of the Ekpyrotic phase of contraction with the beginning of the bounce phase comes from the matching conditions of cosmological perturbations [51, 65, 66]. These conditions impose that the gravitational potential $\Phi_k(\eta)$ and the modified curvature perturbation variable $\hat{\zeta}_k(\eta)$ are continuous across any transition (e.g. from the Ekpyrotic phase of contraction to the bounce phase). The variable $\hat{\zeta}_k$ is defined as [51]

$$\hat{\zeta}_k \equiv \zeta_k + \frac{1}{3}c_s^2 \left(\frac{k}{\mathcal{H}} \right)^2 \Phi_k \left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \right)^{-1}. \quad (78)$$

On super-Hubble scales ($k \ll \mathcal{H}$), we note that the second term of the above expression is suppressed, so $\hat{\zeta}_k \simeq \zeta_k$. Thus, ζ_k must also be continuous across a transition. That is why the values of ζ_k and $\dot{\zeta}_k$ at the end of the Ekpyrotic phase of contraction are taken as the initial conditions of the bounce phase.

At this point, we note that the maximal growth rate for ζ is given by

$$\dot{\zeta}_{\text{max}} \simeq \dot{\zeta}(t_{B-}) \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\text{amp}-})} \right)^2, \quad (79)$$

and that ζ grows at most linearly in time. Therefore, one can say that

$$\zeta(t_{\text{amp}+}) - \zeta(t_{\text{amp}-}) \lesssim \dot{\zeta}(t_{B-}) \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\text{amp}-})} \right)^2 (t_{\text{amp}+} - t_{\text{amp}-}). \quad (80)$$

Furthermore, since ζ receives essentially no amplification outside the interval $[t_{\text{amp}-}, t_{\text{amp}+}]$, we can place an upper bound on the total growth that curvature perturbations on super-Hubble scales receive from the bounce phase,

$$\frac{\Delta\zeta}{\zeta(t_{B-})} \equiv \frac{\zeta(t_{B+}) - \zeta(t_{B-})}{\zeta(t_{B-})} \lesssim \frac{\dot{\zeta}(t_{B-})}{\zeta(t_{B-})} \left(\frac{\dot{\phi}_B}{\dot{\phi}(t_{\text{amp}-})} \right)^2 2\Delta t_{\text{amp}}, \quad (81)$$

where we divide the growth $\Delta\zeta$ by the initial size of ζ before the bounce to get a dimensionless quantity.

E. Comparison with tensor modes

We recall the equation of motion for tensor modes given by Eq. (12), which in the IR limit on super-Hubble scales reduces to

$$h'' + 2\frac{a'}{a}h' = 0. \quad (82)$$

³ If ζ enters the bounce phase with a vanishingly small time derivative, then curvature perturbations will remain constant throughout and exit the bounce phase unaffected.

Once again, we drop the k index when it is clear that the modes are in the IR limit. Close to the bounce point, we recall that the scale factor is almost constant, i.e. $a(\eta) \simeq a_B$. Thus, we are left with the equation $h'' \simeq 0$, and consequently,

$$h(\eta) \simeq h(\eta_i) + h'(\eta_i)(\eta - \eta_i), \quad (83)$$

or, equivalently,

$$h(t) \simeq h(t_i) + \dot{h}(t_i)(t - t_i). \quad (84)$$

Thus, as in the case of curvature fluctuations in Region II in the vicinity of the bounce point, there is a linearly growing mode. Dimensional analysis, however, tells us that this growing mode will not overwhelm the constant mode. The argument is as follows: we can estimate $\dot{h}(t_i)$ to be of the order $Mh(t_i)$, where M is the mass scale at the bounce. On the other hand, we expect the time interval of the bounce phase to be of the order M^{-1} , and hence we expect the linearly growing term to be comparable at the end of the bounce phase to the constant mode.

Comparing the coefficients of the linearly growing modes of the curvature fluctuations and the tensor modes, i.e. Eq. (79) and the coefficient of the growing mode in Eq. (84), respectively, we see that it is the extra factor of $[\dot{\phi}_B/\dot{\phi}(t_{\text{amp}}-)]^2$ in the coefficient of the scalar modes which leads to the enhancement of the scalar power spectrum relative to the tensor power spectrum.

VI. A COMPREHENSIVE ANALYSIS OF THE PRODUCTION OF PRIMORDIAL NON-GAUSSIANITIES DURING THE BOUNCE PHASE

Now that we have identified the conditions under which the tensor-to-scalar ratio can be suppressed, we turn to the study of how the bispectrum evolves during the bounce phase. We make use of the formalism developed in [49] (see also [67, 68]).

Our starting point is the expression (36) for the three-point function. From this expression it is clear that the bispectrum builds up over time, which is to say that the three-point function after the bounce equals the three-point function before the bounce plus the result of integrating the right-hand side of (36) over the time interval of the bounce. From the form (38) of the interaction Lagrangian it follows that the terms which dominate the three-point function in the infrared are given by three powers of ζ and two powers of its time derivative. As shown explicitly in [30] in the computation of the three-point function in the matter-dominated contracting phase, the absolute amplitude of ζ cancels out in the definition of the shape function. Furthermore, Cai *et al.* [30] show that the bispectrum at the end of the period of matter contraction has an amplitude of the order 1 with a shape which is different from what is obtained in simple inflationary models. Since the dominant mode of ζ is constant during the Ekpyrotic phase of contraction, no additional contribution to the bispectrum is generated during that phase. We have not computed the contribution generated during a possible radiation phase of contraction between the end of the matter period and beginning of the Ekpyrotic period. This calculation could be done using the methods of [30] and we would find again a contribution with amplitude of the order of one and with a shape similar to that generated in the matter phase of contraction and different from that in simple inflationary models, the reason being that the same terms which dominate the bispectrum in the matter phase will also dominate in the radiation phase, and they are terms which are slow-roll suppressed during inflation.

Hence, we now turn to the evaluation of the contribution of the bouncing phase to the three-point function. However, we must keep in mind that the equations of [49], in particular the third order perturbed Lagrangian given by Eq. (38), are only valid for a canonical scalar field. We must generalize the analysis to the case of the matter Lagrangian studied here (this generalization will not affect the evolution of the three-point function outside of the bounce phase because the extra terms which we derive below are negligible except in the bounce phase). This has already been done in the case of inflation for very general Lagrangians (see, e.g., [69, 70]).

For the Lagrangian given by Eq. (46), perturbations up to third order in ζ yield the action

$$\begin{aligned} S_3 = \int d^4x \left(B_1 [\partial\zeta\partial\chi\partial^2\zeta - \zeta\partial_i\partial_j(\partial_i\zeta\partial_j\chi)] + B_2\dot{\zeta}^2\partial^2\zeta \right. \\ \left. + B_3\dot{\zeta}\partial\zeta\partial\chi + B_4\zeta(\partial_i\partial_j\chi)^2 + B_5\zeta(\partial\zeta)^2 + B_6\dot{\zeta}^3 + B_7\zeta\dot{\zeta}^2 - 2f(\zeta)\frac{\delta\mathcal{L}_2}{\delta\zeta} \right), \end{aligned} \quad (85)$$

where

$$f(\zeta) = \frac{A_{20}a^2}{4M_p^2} [(\partial\zeta)^2 - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\chi)] + \frac{A_{18}a^2}{M_p^2} [\partial\zeta\partial\chi - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\chi)] - \frac{2A_4a^3 - C_1}{2z^2c_s^2} a\zeta\dot{\zeta}. \quad (86)$$

The derivation of this action and the form of the functions A_n , B_n , and C_n ($n = 1, \dots$) can be found in Appendix C1. As expected, this action is equivalent to the action given by Eq. (38) in the limit where the Lagrangian (46) is canonical in a matter-dominated contracting universe. This is shown in Appendix C2.

In order to cancel the last term in Eq. (85), we make a field redefinition in Fourier space $\zeta(\eta, \vec{k}) \rightarrow \zeta(\eta, \vec{k}) - f(\eta, \vec{k})$ in the third order Lagrangian. This way, there will be two contributions to the three-point function. The first part of the three-point function is the third order Lagrangian without the last term and the second part is related to the field redefinition terms where $\zeta(\eta, \vec{k})$ is replaced by $f(\eta, \vec{k})$. Using the Lagrangian formalism, we note that in Fourier space, we can canonically express the modes $\zeta(\eta, \vec{k})$ as follows,

$$\zeta(\eta, \vec{k}) = \zeta_k(\eta) a_k^\dagger + \zeta_k^*(\eta) a_{-\vec{k}} , \quad (87)$$

where $a_{\vec{k}}|0\rangle = 0$, so $a_{\vec{k}}$ is the annihilation operator, and $a_{\vec{k}}^\dagger$ is the respective creation operator. Then, if we consider the interaction picture, the three-point function to leading order in the interaction coupling constant is given by

$$\langle \zeta(\eta, \vec{k}_1) \zeta(\eta, \vec{k}_2) \zeta(\eta, \vec{k}_3) \rangle_{\text{int}} = i \int_{\eta_i}^{\eta} d\tilde{\eta} \langle [\zeta(\eta, \vec{k}_1) \zeta(\eta, \vec{k}_2) \zeta(\eta, \vec{k}_3), L_{\text{int}}(\tilde{\eta})] \rangle , \quad (88)$$

where η_i corresponds to the initial time before which there is no non-Gaussianity. Also, L_{int} is associated with the third order action (85) without its last term.

Here, we are interested in the production of non-Gaussianities during the bounce phase, so we consider the initial time to be the beginning of the bounce phase and we consider the end time at which the three-point function is evaluated to be the end of the bounce phase. However, as we saw in the previous section, curvature perturbations are nearly constant, and hence do not contribute to the three-point function, except during the small time interval $[\eta_{\text{amp-}}, \eta_{\text{amp+}}]$ where ζ grows. Thus, the integration bounds are taken to be from $\eta_{\text{amp-}}$ to $\eta_{\text{amp+}}$, and the evolution of the curvature perturbations is taken to be

$$\zeta_k(\eta) = \zeta_k^{\text{m}}(\eta_{B-}) + \zeta_k^{\text{m}'}(\eta_{B-}) \left(\frac{\phi_B'}{\phi'(\eta_{\text{amp-}})} \right)^2 (\eta - \eta_{\text{amp-}}) . \quad (89)$$

The above expression follows from taking the maximal linear growth rate given by Eq. (79) throughout the amplification interval $[\eta_{\text{amp-}}, \eta_{\text{amp+}}]$. This expression slightly underestimates ζ_k for $\eta_{\text{amp-}} \leq \eta < \eta_B$ and slightly overestimates ζ_k for $\eta_B < \eta \leq \eta_{\text{amp+}}$ but it is a good approximation on average over the small interval $[\eta_{\text{amp-}}, \eta_{\text{amp+}}]$.

We recall that curvature perturbations are more or less constant during the Ekpyrotic phase of contraction that precedes the bounce phase. Therefore, it is natural to take the end conditions of the matter-dominated phase of contraction as the initial conditions of the bounce phase. As shown in Sec. VD, ζ_k and ζ_k' must be continuous across any transition on super-Hubble scales. Hence for the initial conditions of the bounce phase, we put the superscript “m” which denotes the matter bounce solution [30]

$$\zeta_k^{\text{m}}(\eta) = \frac{i A e^{ik(\eta - \tilde{\eta}_{B-})} [1 - ik(\eta - \tilde{\eta}_{B-})]}{\sqrt{2} k^3 (\eta - \tilde{\eta}_{B-})^3} , \quad (90)$$

where $\tilde{\eta}_{B-}$ is the conformal time at the singularity if the matter-dominated contracting phase were to continue to arbitrary densities (i.e. without NEC violating matter). Also, A is a normalization constant which is determined from the quantum vacuum condition at Hubble radius crossing in the contracting phase, and it is found to be

$$A = \frac{(\Delta \eta_{B-})^2}{\sqrt{3} a_B M_p} , \quad (91)$$

where $\Delta \eta_{B-} \equiv \tilde{\eta}_{B-} - \eta_{B-}$.

Let us comment on the wave number dependence of Eq. (89). We first solved the equation of motion in the bouncing phase in the limit where $k \ll \mathcal{H}$ to 0th order. Then, matching the solution at the beginning of the bouncing phase with the one at the end of the matter contraction phase, we introduced some wave number dependence since the solution in the matter contraction phase has higher order terms in k/\mathcal{H} . Thus, one may worry that obtaining the correct k -dependent solution in the bounce phase up to leading order requires one to solve the full k -dependent equation of motion. However, we note that we will be interested in the IR limit again when evaluating the three-point function. Thus, any k dependence not included in the above solution is suppressed during the bounce phase as long as k remains much smaller than the largest energy scale attained during the bounce, i.e. as long as $k \ll \mathcal{H}_{B-}, \mathcal{H}_{B+}$, and as long as the corresponding wavelength of the fluctuations remains much larger than the bounce length scale, which can be reformulated as $k \ll (\Delta \eta_B)^{-1}$.

Substituting the interaction Lagrangian L_{int} associated with the action (85) without its last term into Eq. (88) and using Eq. (87), we find

$$\begin{aligned}
\langle \zeta(\vec{k}_1) \zeta(\vec{k}_2) \zeta(\vec{k}_3) \rangle_{\text{int}} &= (2\pi)^3 \delta^{(3)} \left(\sum_{i=1}^3 \vec{k}_i \right) \zeta_{k_1}^*(\eta_+) \zeta_{k_2}^*(\eta_+) \zeta_{k_3}^*(\eta_+) \\
&\times i \int_{\eta_-}^{\eta_+} d\eta \left[\frac{B_1(\eta) z^2(\eta)}{M_p^2} \left(\frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} k_2^2 + \frac{[\vec{k}_2 \cdot (\vec{k}_2 - \vec{k}_3)][\vec{k}_3 \cdot (\vec{k}_2 + \vec{k}_3)]}{k_3^2} \right) \zeta_{k_1}(\eta) \zeta_{k_2}(\eta) \zeta'_{k_3}(\eta) \right. \\
&+ \left(\frac{B_2(\eta)}{a_B} k_1^2 + \frac{B_3(\eta) z^2(\eta)}{M_p^2 a_B} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta) z^4(\eta)}{M_p^4 a_B} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + \frac{B_7(\eta)}{a_B} \right) \zeta_{k_1}(\eta) \zeta'_{k_2}(\eta) \zeta'_{k_3}(\eta) \\
&+ B_5(\eta) a_B (\vec{k}_1 \cdot \vec{k}_2) \zeta_{k_1}(\eta) \zeta_{k_2}(\eta) \zeta_{k_3}(\eta) + \left. \left(\frac{B_6(\eta)}{a_B^2} \right) \zeta'_{k_1}(\eta) \zeta'_{k_2}(\eta) \zeta'_{k_3}(\eta) \right] + (5 \text{ permutations}) .
\end{aligned} \tag{92}$$

Moreover, the contribution from the field redefinition is

$$\begin{aligned}
-\langle \zeta(\vec{k}_1) \zeta(\vec{k}_2) f(\vec{k}_3) \rangle_{\text{redef}} &= (2\pi)^3 \delta^{(3)} \left(\sum_{i=1}^3 \vec{k}_i \right) \\
&\times \left[\frac{A_{20}(\eta_+) a_B^2}{4M_p^2} \left(-\vec{k}_1 \cdot (\vec{k}_3 - \vec{k}_1) + \frac{(\vec{k}_1 \cdot \vec{k}_3)[(\vec{k}_3 - \vec{k}_1) \cdot \vec{k}_3]}{k_3^2} \right) |\zeta_{k_1}(\eta_+)|^2 |\zeta_{k_2}(\eta_+)|^2 \right. \\
&- \frac{A_{18}(\eta_+) a_B z^2(\eta_+)}{M_p^4} \left(\frac{\vec{k}_1 \cdot (\vec{k}_3 - \vec{k}_1)}{k_1^2} - \frac{(\vec{k}_1 \cdot \vec{k}_3)[(\vec{k}_3 - \vec{k}_1) \cdot \vec{k}_3]}{k_1^2 k_3^2} \right) \zeta'_{k_1}(\eta_+) \zeta_{k_1}^*(\eta_+) |\zeta_{k_2}(\eta_+)|^2 \\
&+ \left. \left(\frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{2z^2(\eta_+) c_s^2} \right) \zeta'_{k_1}(\eta_+) \zeta_{k_1}^*(\eta_+) |\zeta_{k_2}(\eta_+)|^2 \right] + (5 \text{ permutations}) .
\end{aligned} \tag{93}$$

The permutations that we refer to are over the \vec{k}_i vectors for $i = 1, 2, 3$. We note that, to simplify the notation, we set $\eta_{\pm} \equiv \eta_{\text{amp}\pm}$. The general form of the full three-point function can be expressed as

$$\langle \zeta(\vec{k}_1) \zeta(\vec{k}_2) \zeta(\vec{k}_3) \rangle = \langle \zeta(\vec{k}_1) \zeta(\vec{k}_2) \zeta(\vec{k}_3) \rangle_{\text{int}} + \langle \zeta(\vec{k}_1) \zeta(\vec{k}_2) f(\vec{k}_3) \rangle_{\text{redef}} = (2\pi)^7 \delta^{(3)} \left(\sum_i \vec{k}_i \right) \frac{\mathcal{P}_{\zeta}^2}{\prod_i k_i^3} \mathcal{A}(k_1, k_2, k_3) , \tag{94}$$

and so, if we substitute Eqs. (92) and (93) into the above, we find the shape function to be given by

$$\begin{aligned}
\mathcal{A}(k_1, k_2, k_3) &= \frac{k_3^3 \zeta_{k_1}^*(\eta_+) \zeta_{k_2}^*(\eta_+) \zeta_{k_3}^*(\eta_+)}{4 \zeta_{k_1}^*(\eta_+) \zeta_{k_2}^*(\eta_+) \zeta_{k_1}(\eta_+) \zeta_{k_2}(\eta_+)} \\
&\times i \int_{\eta_-}^{\eta_+} d\eta \left[B_1(\eta) z^2(\eta) \left(\frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} k_2^2 - \frac{[\vec{k}_2 \cdot (\vec{k}_2 + \vec{k}_3)][\vec{k}_3 \cdot (\vec{k}_2 + \vec{k}_3)]}{k_3^2} \right) \zeta_{k_1}(\eta) \zeta_{k_2}(\eta) \zeta'_{k_3}(\eta) \right. \\
&+ \left(\frac{B_2(\eta)}{a_B} k_1^2 + \frac{B_3(\eta) z^2(\eta)}{M_p^2 a_B} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta) z^4(\eta)}{M_p^4 a_B} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + \frac{B_7(\eta)}{a_B} \right) \zeta_{k_1}(\eta) \zeta'_{k_2}(\eta) \zeta'_{k_3}(\eta) \\
&+ B_5(\eta) a_B (\vec{k}_1 \cdot \vec{k}_2) \zeta_{k_1}(\eta) \zeta_{k_2}(\eta) \zeta_{k_3}(\eta) + \frac{B_6(\eta)}{a_B^2} \zeta'_{k_1}(\eta) \zeta'_{k_2}(\eta) \zeta'_{k_3}(\eta) \Big] \\
&+ \frac{A_{20}(\eta_+) a_B^2 k_3^3}{4M_p^2} \left(-\vec{k}_1 \cdot (\vec{k}_3 - \vec{k}_1) + \frac{(\vec{k}_1 \cdot \vec{k}_3)[(\vec{k}_3 - \vec{k}_1) \cdot \vec{k}_3]}{k_3^2} \right) \\
&- \frac{A_{18}(\eta_+) a_B z^2(\eta_+) k_3^3}{M_p^4} \left(\frac{\vec{k}_1 \cdot (\vec{k}_3 - \vec{k}_1)}{k_1^2} - \frac{(\vec{k}_1 \cdot \vec{k}_3)[(\vec{k}_3 - \vec{k}_1) \cdot \vec{k}_3]}{k_1^2 k_3^2} \right) \frac{\zeta'_{k_1}(\eta_+) \zeta_{k_1}^*(\eta_+)}{|\zeta_{k_1}(\eta_+)|^2} \\
&+ k_3^3 \left(\frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{2z^2(\eta_+) c_s^2} \right) \frac{\zeta'_{k_1}(\eta_+) \zeta_{k_1}^*(\eta_+)}{|\zeta_{k_1}(\eta_+)|^2} + (5 \text{ permutations}) .
\end{aligned} \tag{95}$$

At this point, we should note that the contributions coming from the terms with coefficients B_1 , B_2 , B_5 , and A_{20} are of order $\mathcal{O}(k^5)$, and consequently, these terms are vanishingly small compared to other terms, which are of order

$\mathcal{O}(k^3)$, on super-Hubble scales. Therefore, the three main contributions to the shape function are the $\zeta\zeta'^2$ term, the ζ'^3 term, and the field redefinition term. We evaluate each of these terms separately in Appendix D and we find the general expression for the shape function after the bounce phase [see Eq. (D18)].

Three important forms of non-Gaussianity in cosmological observations are the local form, the equilateral form, and the orthogonal form. The local form of non-Gaussianity requires that one of the three momentum modes exits the Hubble radius much earlier than the other two, i.e. $k_1 \ll k_2 = k_3$. Evaluating the shape function (D18) in this limit yields

$$\begin{aligned} f_{\text{NL}}^{\text{local}} &= \frac{10}{3} \frac{\mathcal{A}(k_1 \ll k_2 = k_3)}{\sum_i k_i^3} \\ &\simeq \frac{10}{3} \left[\frac{-A^2}{8a_B \Delta\eta_{B-}^4} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \left(\frac{B_4(\eta_B) z_B^4}{M_p^4} + B_7(\eta_B) \right) - \frac{3A^2}{8\Delta\eta_{B-}^4 \Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \frac{B_6(\eta_B)}{a_B^2} \right. \\ &\quad \left. + \frac{A_{18}(\eta_+) a_B z^2(\eta_+)}{8M_p^4 \Delta\eta_{\text{amp}}} + \frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{8z^2(\eta_+) c_s^2 \Delta\eta_{\text{amp}}} \right]. \end{aligned} \quad (96)$$

The equilateral form of non-Gaussianity requires that $k_1 = k_2 = k_3$, so one finds

$$\begin{aligned} f_{\text{NL}}^{\text{equil}} &= \frac{10}{3} \frac{\mathcal{A}(k_1 = k_2 = k_3)}{\sum_i k_i^3} \\ &\simeq \frac{10}{3} \left[\frac{A^2}{16a_B \Delta\eta_{B-}^4} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \left(\frac{B_3(\eta_B) z_B^2}{M_p^2} - \frac{B_4(\eta_B) z_B^4}{2M_p^4} - 2B_7(\eta_B) \right) - \frac{3A^2}{8\Delta\eta_{B-}^4 \Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \frac{B_6(\eta_B)}{a_B^2} \right. \\ &\quad \left. + \frac{3A_{18}(\eta_+) a_B z^2(\eta_+)}{16M_p^4 \Delta\eta_{\text{amp}}} + \frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{8z^2(\eta_+) c_s^2 \Delta\eta_{\text{amp}}} \right]. \end{aligned} \quad (97)$$

Finally, the orthogonal form of non-Gaussianity requires that $k_1 = \sqrt{k_2^2 + k_3^2} = \sqrt{2}k$, so one finds

$$\begin{aligned} f_{\text{NL}}^{\text{ortho}} &= \frac{10}{3} \frac{\mathcal{A}(k_1 = \sqrt{k_2^2 + k_3^2} = \sqrt{2}k)}{\sum_i k_i^3} \\ &\simeq \frac{10}{3} \left[\frac{A^2}{16a_B \Delta\eta_{B-}^4} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \left((4 - 3\sqrt{2}) \frac{B_3(\eta_B) z_B^2}{M_p^2} + (4 - 2\sqrt{2}) \frac{B_4(\eta_B) z_B^4}{M_p^4} - 2B_7(\eta_B) \right) \right. \\ &\quad \left. - \frac{3A^2}{8\Delta\eta_{B-}^4 \Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \frac{B_6(\eta_B)}{a_B^2} + (1 + \sqrt{2}) \frac{A_{18}(\eta_+) a_B z^2(\eta_+)}{16M_p^4 \Delta\eta_{\text{amp}}} + \frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{8z^2(\eta_+) c_s^2 \Delta\eta_{\text{amp}}} \right]. \end{aligned} \quad (98)$$

Substituting in some values for the model parameters (Υ , T , ϕ'_B , γ , etc., introduced in Sec. IV) would yield specific numbers for the amount of non-Gaussianities that has been produced during the bounce. However, instead of giving exact values now, we will try to constrain the parameter space from observations. This is what we do in the next section.

VII. COMBINATION OF THE OBSERVATIONAL BOUNDS ON NON-GAUSSIANITIES AND ON THE TENSOR-TO-SCALAR RATIO

Let first rewrite the expression for $f_{\text{NL}}^{\text{local}}$ using Eqs. (91), (D9), (D10), (D12), (D16), and (D17),

$$\begin{aligned} f_{\text{NL}}^{\text{local}} &\simeq - \frac{5}{12\gamma^4 \phi_B'^8} (3a_B^2 \beta \gamma^2 M_p^2 \phi_B'^6 + 3\gamma^4 \phi_B'^8 + 12a_B^8 \beta M_p^6 \Upsilon + 16a_B^6 \gamma^2 M_p^4 \phi_B'^2 \Upsilon) \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \\ &\quad - \frac{25a_B^2 M_p^2}{6\gamma^3 \phi_B'^5} (2a_B^2 \beta M_p^2 + 3\gamma^2 \phi_B'^2) \frac{1}{\Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 + \frac{5a_B^4 \beta M_p^4}{4\gamma^3 \phi_B'^5} \frac{1}{\Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^5 \\ &\quad + \frac{5a_B^2 M_p^2}{\gamma \phi_B'^3} \frac{1}{\Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^3 + \frac{10\gamma}{3\beta \phi_B'} \frac{1}{\Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right). \end{aligned} \quad (99)$$

The equilateral and orthogonal f_{NL} have similar expressions, only with different coefficients.

At this point, we do not want to insert specific values for the model parameters. Yet, in order to have a healthy bounce, i.e. one that yields a bounce free of ghost instabilities, we expect the model parameters to lie in specific regimes. From [22, 23, 28], we expect that $\phi'_B < a_B M_p$, $\Upsilon \ll M_p^2$, $\beta \sim \mathcal{O}(1)$, and $\gamma \ll 1$. Also, from Eq. (54), it is obvious that $\phi'_B/\phi'(\eta_-) > 1$. Therefore, keeping only dominant terms, the expression for $f_{\text{NL}}^{\text{local}}$ reduces to

$$f_{\text{NL}}^{\text{local}} \simeq \frac{5a_B^4 \beta M_p^4}{\gamma^3 \phi_B'^5} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \left[-\frac{a_B^4 M_p^2 \Upsilon}{\gamma \phi_B'^3} - \frac{5}{3\Delta\eta_{\text{amp}}} + \frac{1}{4\Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^3 \right]. \quad (100)$$

In the square bracket, the three terms come from $\mathcal{A}_{\zeta\zeta'^2}$, $\mathcal{A}_{\zeta'^3}$, and $\mathcal{A}_{\text{redef}}$, respectively. However, let us recall from Appendix D that the results for $\mathcal{A}_{\zeta\zeta'^2}$ and $\mathcal{A}_{\zeta'^3}$ were actually upper bounds in absolute value. Since we expect that $\Delta\eta_{\text{amp}} \sim \mathcal{O}(1/a_B M_p)$ from Eq. (70), it results that the field redefinition term is dominant over the other ones, just like in the regular matter bounce [30], so we can write

$$f_{\text{NL}}^{\text{local}} \simeq \frac{5a_B^4 \beta M_p^4}{4\gamma^3 \phi_B'^5} \frac{1}{\Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^5. \quad (101)$$

In order to combine the bound on curvature perturbations and the above result, it is useful to rewrite the expressions for f_{NL} in terms of the ratio $\Delta\zeta/\zeta(\eta_{B-})$. In Sec. V, Eq. (81) told us that

$$\frac{\Delta\zeta}{\zeta(\eta_{B-})} \lesssim 2 \frac{\zeta'(\eta_{B-})}{\zeta(\eta_{B-})} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \Delta\eta_{\text{amp}}. \quad (102)$$

In the previous section, we argued that the initial conditions at η_{B-} were given by the end conditions of the matter-dominated phase of contraction, so we can say that

$$\zeta(\eta_{B-}) \stackrel{k/\mathcal{H} \rightarrow 0}{\simeq} \zeta_k^{\text{m}}(\eta_{B-}), \quad \zeta'(\eta_{B-}) \stackrel{k/\mathcal{H} \rightarrow 0}{\simeq} \zeta_k^{\text{m}'}(\eta_{B-}). \quad (103)$$

Recalling that ζ_k^{m} is given by Eq. (90), we find that

$$\frac{\zeta'(\eta_{B-})}{\zeta(\eta_{B-})} \simeq \lim_{k/\mathcal{H} \rightarrow 0} \frac{\zeta_k^{\text{m}'}(\eta_{B-})}{\zeta_k^{\text{m}}(\eta_{B-})} = \frac{3}{\Delta\eta_{B-}}, \quad (104)$$

and thus Eq. (102) becomes

$$\frac{1}{6} \left(\frac{\Delta\eta_{B-}}{\Delta\eta_{\text{amp}}} \right) \left(\frac{\Delta\zeta}{\zeta(\eta_{B-})} \right) \lesssim \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2. \quad (105)$$

This allows us to place a lower bound on Eq. (101) as follows,

$$f_{\text{NL}}^{\text{local}} \gtrsim \frac{5a_B^4 \beta M_p^4}{144\sqrt{6}\gamma^3 \phi_B'^5} \frac{1}{\Delta\eta_{\text{amp}}} \left(\frac{\Delta\eta_{B-}}{\Delta\eta_{\text{amp}}} \right)^{5/2} \left(\frac{\Delta\zeta}{\zeta(\eta_{B-})} \right)^{5/2}. \quad (106)$$

Our initial estimation in Sec. III C showed that we expected f_{NL} to have terms of order $(\Delta\zeta/\zeta)^1$ and $(\Delta\zeta/\zeta)^2$. The terms of order $(\Delta\zeta/\zeta)^1$ in the full calculation corresponded to terms of order $[\phi'_B/\phi'(\eta_-)]^2$ in our approximation scheme and they originated from $\mathcal{A}_{\zeta\zeta'^2}$ and $\mathcal{A}_{\zeta'^3}$. A term of order $(\Delta\zeta/\zeta)^2$, i.e. of order $[\phi'_B/\phi'(\eta_-)]^4$, could have originated from $\mathcal{A}_{\zeta\zeta'^2}$ but the full calculation showed that it did not have any real component [see Eq. (D5)]. Instead, the full calculation showed the presence of terms of order $(\Delta\zeta/\zeta)^{5/2}$, $(\Delta\zeta/\zeta)^{3/2}$, and $(\Delta\zeta/\zeta)^{1/2}$ coming from the field redefinition contribution to the shape function. In the large amplification limit, we are left with one term of order $(\Delta\zeta/\zeta)^{5/2}$ as shown in Eq. (106).

Let us recall that $\Delta\zeta/\zeta$ is bounded from below in order to satisfy the current observational bound on the tensor-to-scalar ratio r . Using the bound (31), we can further constrain the bound (106),

$$f_{\text{NL}}^{\text{local}} \gtrsim 240 \left(\frac{\beta}{\gamma^3} \right) \left(\frac{a_B M_p}{\phi'_B} \right)^5 \left(\frac{(a_B M_p)^{-1}}{\Delta\eta_{\text{amp}}} \right) \left(\frac{\Delta\eta_{B-}}{\Delta\eta_{\text{amp}}} \right)^{5/2}. \quad (107)$$

Let us note that $\Delta\eta_{B-} \sim \mathcal{H}_{B-}^{-1}$, and since the bounce energy scale is taken to be much less than the Planck scale, it results that $\Delta\eta_{B-} \gg (a_B M_p)^{-1}$. Thus, since every dimensionless ratio in Eq. (107) at least of order 1 or much greater than 1, it results that $f_{\text{NL}}^{\text{local}} \gtrsim 240$. Including the negative contribution to $f_{\text{NL}}^{\text{local}}$ from the matter-dominated

contracting phase which is of order 1 [30] and the negative contributions from $\mathcal{A}_{\zeta'^2}$ and $\mathcal{A}_{\zeta'^3}$ would reduce this bound, but really not significantly.

The best observational bounds on primordial non-Gaussianities currently come from the Planck experiment, which reports [32]

$$f_{\text{NL}}^{\text{local}} = 0.8 \pm 5.0, \quad f_{\text{NL}}^{\text{equil}} = -4 \pm 43, \quad f_{\text{NL}}^{\text{ortho}} = -26 \pm 21, \quad (108)$$

at 68% CL. We see that the lower bound on $f_{\text{NL}}^{\text{local}}$ coming from the bounce phase is already excluded by the observations at very high confidence level. Following the same analysis as above for the equilateral and orthogonal shapes yields the bounds $f_{\text{NL}}^{\text{equil}} \gtrsim 359$ and $f_{\text{NL}}^{\text{ortho}} \gtrsim 289$, which are also excluded at very high confidence level, although to a smaller extent than $f_{\text{NL}}^{\text{local}}$.

To summarize, in this section we took the non-Gaussianity results derived in the previous section and imposed that there had been a sufficient amplification of curvature perturbations in order to satisfy the current observational bound on the tensor-to-scalar ratio. As a result, the theoretical lower bounds on $f_{\text{NL}}^{\text{local}}$, $f_{\text{NL}}^{\text{equil}}$, and $f_{\text{NL}}^{\text{ortho}}$ are excluded at high confidence level by the current observational constraints on non-Gaussianities. This shows that the model suffers from the “no-go” theorem that we conjectured in Sec. III D.

Looking at Eq. (107), we see that this could be alleviated if, for instance, the amplification period was very long compared to the Planck time, or if the model parameters were such that $\beta/\gamma^3 \ll 1$ or $a_B M_p/\phi'_B \ll 1$. However, these conditions seem unlikely to occur in a physically admissible matter bounce scenario.

VIII. CONCLUSIONS

In the present paper, we have studied in detail the nonlinear dynamics of primordial curvature perturbations during the nonsingular bouncing phase in a matter bounce model described by a single generic scalar field minimally coupled to Einstein gravity. This type of model can be made consistent with the observational bound on the tensor-to-scalar ratio by realizing an enhancement of the curvature perturbations in the bouncing phase. We derived the conditions on the model parameters for which such an enhancement can be achieved. We then expanded the action for perturbations up to the third order, computed the full set of three point correlation functions and then derived the nonlinearity parameters f_{NL} in the cases of specific shapes of observational interest. Our results show that if the primordial non-Gaussianities mainly arise from a manifest growth of curvature perturbations during the bounce, then the nonlinearity parameter would become dangerously large and lead to a disagreement with the observational constraints from cosmic microwave background (CMB) data⁴. Specifically, we examined the relation between the nonlinearity parameter in the local, equilateral, and orthogonal limits and the growth of the curvature perturbations and explicitly showed that the observational bounds on the tensor-to-scalar ratio and the CMB bispectrum cannot be simultaneously satisfied. This leads us to conjecture that there is a “no-go” theorem for single field matter bounce cosmologies, assuming that the nonsingular bounce is realized by a generic scalar field minimally coupled to Einstein gravity, *which would rule out a large class of matter bounce models*.

We note that this “no-go” theorem might be circumvented by dropping certain assumptions imposed above. For instance, if one introduces more than one degree of freedom such as in the matter bounce curvaton mechanism [58, 71], the constraints from the tensor-to-scalar ratio as well as from the primordial non-Gaussianities can be satisfied at the same time, the reason being that in the curvaton scenario the scalar fluctuations are generated by a different mechanism than the tensors. As another example, if the initial Bunch-Davies vacuum is noncanonical (e.g., in the Λ CDM bounce [6], the initial quantum vacuum has $c_s \ll 1$), the initial ratio of the tensor modes to the scalar modes can be suppressed, in which case there is no need for the curvature perturbations to be enhanced during the bounce.

Our analysis also does not immediately apply to nonsingular matter bounce models in which the violation of the null energy condition is obtained by changes in the gravitational action (e.g., in Loop Quantum Cosmology [72, 73], Horava-Lifshitz gravity [13], extended $F(R)$ gravity [53, 54], modified Gauss-Bonnet gravity [55], or torsion gravity scenarios [56, 57]). We might expect that the no-go theorem will extend to modified gravity matter bounce models which have the same number of degrees of freedom as General Relativity, in which case the tensor-to-scalar is generically large [28]. However, if the gravity model contains new degrees of freedom, then we might be in a situation similar to what happens in the curvaton scenario: the new degrees of freedom source the scalar modes but not the tensor modes, thus suppressing the tensor-to-scalar ratio during the bounce phase. Yet, it would be interesting to explicitly

⁴ We recall that it has also been found in [33, 34] that non-Gaussianities could become dangerously large in a certain nonsingular bouncing cosmology and it has been conjectured that this could be generic to a large family of nonsingular bouncing cosmologies.

analyze the conditions under which the bispectrum constraints can be made consistent with the observed bound on the tensor-to-scalar ratio in such models.

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Appendix A: CURVATURE PERTURBATIONS EXPANDING ABOUT THE SINGULARITY

The equation of motion for curvature perturbations in the IR limit is [see Eq. (11)]

$$\frac{d\zeta'}{d\eta} + \frac{(z^2)'}{z^2} \zeta' = 0. \quad (\text{A1})$$

Let us parametrize z^2 close to the singular point η_s as

$$z^2(\eta) \sim \frac{1}{(\eta - \eta_s)^2}, \quad (\text{A2})$$

so the equation of motion becomes

$$\frac{d\zeta'}{d\eta} = \frac{2}{\eta - \eta_s} \zeta'. \quad (\text{A3})$$

Since after the singular time we have $\eta > \eta_s > \eta_i$, we integrate as follows,

$$\ln \left(\frac{\zeta'(\eta)}{\zeta'(\eta_i)} \right) = 2 \int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{\tilde{\eta} - \eta_s} = 2 \left(\int_{\eta_i}^{\eta_s - \epsilon} + \int_{\eta_s - \epsilon}^{\eta_s + \epsilon} + \int_{\eta_s + \epsilon}^{\eta} \right) \frac{d\tilde{\eta}}{\tilde{\eta} - \eta_s}, \quad (\text{A4})$$

for some constant ϵ . As we take the limit $\epsilon \rightarrow 0$, the second integral vanishes by definition and we are left with the first and third integral. Evaluating them, we find

$$\begin{aligned} \ln \left(\frac{\zeta'(\eta)}{\zeta'(\eta_i)} \right) &= 2 \lim_{\epsilon \rightarrow 0} \left[\ln \left(\frac{(\eta_s - \epsilon) - \eta_s}{\eta_i - \eta_s} \right) + \ln \left(\frac{\eta - \eta_s}{(\eta_s + \epsilon) - \eta_s} \right) \right] \\ &= 2 \lim_{\epsilon \rightarrow 0} \ln \left(\frac{-\epsilon(\eta - \eta_s)}{(\eta_i - \eta_s)\epsilon} \right) \\ &= 2 \ln \left(\frac{\eta - \eta_s}{\eta_s - \eta_i} \right). \end{aligned} \quad (\text{A5})$$

Therefore,

$$\zeta'(\eta) = \zeta'(\eta_i) \left(\frac{\eta - \eta_s}{\eta_s - \eta_i} \right)^2 \quad (\text{A6})$$

as expected if there were no singularity. A final integration yields

$$\zeta(\eta) = \zeta(\eta_i) + \zeta'(\eta_i) \left(\frac{(\eta - \eta_s)^3 + (\eta_s - \eta_i)^3}{3(\eta_s - \eta_i)^2} \right). \quad (\text{A7})$$

As expected, we recover the constant mode solution $\zeta' = 0$ as $\eta \rightarrow \eta_s$.

Appendix B: PERTURBATIONS OUTSIDE THE BOUNCE PHASE

Let us consider matter with an equation of state $P = w\rho$ with w independent of time. In this case $z(t) \sim a(t)M_p$ (see Sec. III A). Then, the solution to the long wavelength curvature perturbations is given by [see Eq. (72)]

$$\zeta(t) = \zeta(t_i) + \dot{\zeta}(t_i)a(t_i)z(t_i)^2 \int_{t_i}^t \frac{d\tilde{t}}{a(\tilde{t})z^2(\tilde{t})} \quad (\text{B1})$$

$$= \zeta(t_i) + \dot{\zeta}(t_i)a(t_i)^3 \int_{t_i}^t \frac{d\tilde{t}}{a^3(\tilde{t})} . \quad (\text{B2})$$

For a constant $w \neq -1$, the solution to the scale factor is given by

$$a(t) = a_0 t^{2/3(1+w)} , \quad (\text{B3})$$

for some positive constant a_0 , so we find

$$\zeta(t) = \zeta(t_i) + \dot{\zeta}(t_i)t_i^{\frac{2}{1+w}} \int_{t_i}^t d\tilde{t} \tilde{t}^{-\frac{2}{1+w}} \quad (\text{B4})$$

$$= \zeta(t_i) + \dot{\zeta}(t_i)t_i^{\frac{2}{1+w}} \left(\frac{w+1}{w-1} \right) \left(t^{\frac{w-1}{w+1}} - t_i^{\frac{w-1}{w+1}} \right) , \quad (\text{B5})$$

as long as $|w| \neq 1$. Thus, for matter with $|w| > 1$, the solution for ζ exhibits a constant mode and a mode which is growing in an expanding universe (decaying in a contracting background), whereas for matter with $|w| < 1$, it exhibits a constant mode and a mode which is decaying in an expanding universe (and growing in a contracting background). For example, this implies that an Ekpyrotic phase of contraction in which $w \gg 1$ has a constant mode and a decaying mode.

For $w = -1$, one would recover the standard result of inflation where the constant mode is dominant on super-Hubble scales in an expanding background, and the second mode dominates in a contracting space.

The $w = 1$ case of fast roll expansion is relevant for the dynamics of our nonsingular bouncing cosmology right after the bounce phase. A phase of fast roll expansion occurs if the Lagrangian for the scalar field is dominated by its kinetic term, i.e. $V(\phi) \ll \dot{\phi}^2/2$. It then follows that the solution for the curvature perturbations in this case is (here in conformal time)

$$\zeta(\eta) = \zeta(\eta_i) + \zeta'(\eta_i)a(\eta_i)^2 \int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{a^2(\tilde{\eta})} . \quad (\text{B6})$$

Solving the background dynamics tells us that the solution to the scale factor in a phase of fast roll expansion is

$$a(\eta) = c_E(\eta - \eta_E)^{1/2} , \quad (\text{B7})$$

where c_E and η_E are some constants. Thus,

$$\zeta(\eta) = \zeta(\eta_i) + \zeta'(\eta_i)(\eta_i - \eta_E) \int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{\eta - \eta_E} \quad (\text{B8})$$

$$= \zeta(\eta_i) + \zeta'(\eta_i)(\eta_i - \eta_E) \ln \left(\frac{\eta - \eta_E}{\eta_i - \eta_E} \right) . \quad (\text{B9})$$

So, for $w = 1$, curvature perturbations exhibit a constant mode solution and a logarithmically growing mode, i.e. $\zeta(\eta) \sim \ln \eta$. We note that this is also true in physical time since $a \sim t^{1/3} \sim \eta^{1/2}$ implies that $\zeta(t) \sim \ln t^{2/3} \sim \ln t$.

Appendix C: THIRD ORDER PERTURBED ACTION

1. Derivation of the general form of the third order action

To study the three point correlation function in this matter bounce model, we have to evaluate the action up to third order in perturbation theory. We use the metric in the Arnowitt-Deser-Misner (ADM) form (see, e.g., [68])

$$ds^2 = N^2 dt^2 - \gamma_{ij}(N^i dt + dx^i)(N^j dt + dx^j) , \quad (\text{C1})$$

where $N_i = \gamma_{ij}N^j$ is the shift vector and N is the lapse function. The tensor γ_{ij} is the metric of the 3-dimensional spacelike hypersurfaces in this 3 + 1 decomposition. It is related to the full 4-dimensional space-time metric tensor $g_{\mu\nu}$ via $\sqrt{-g} = N\sqrt{\gamma}$, where g and γ are the determinants of the tensors $g_{\mu\nu}$ and γ_{ij} , respectively. The action (45) in this ADM decomposition is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} \left(R^{(3)} + \kappa_{ij}\kappa^{ij} - \kappa^2 \right) + K(\phi, X) + G(\phi, X)\square\phi \right], \quad (C2)$$

where $R^{(3)}$ is the three-dimensional Ricci scalar and the extrinsic curvature is defined by

$$\kappa_{ij} \equiv \frac{1}{2N} (\dot{\gamma}_{ij} - D_i N_j - D_j N_i). \quad (C3)$$

We define the covariant derivative D_i on the spacelike hypersurfaces such that it is torsion-free and satisfies

$$D_i \gamma^{ij} = 0. \quad (C4)$$

Then, $R^{(3)}_{jkl}$ is the Riemann tensor associated with this connection, and

$$R^{(3)}_{ij} = R^{(3)}_{ikj}, \quad (C5)$$

$$R^{(3)} = \gamma^{ij} R^{(3)}_{ij}, \quad (C6)$$

are the Ricci tensor and Ricci scalar, respectively. In the uniform field gauge where

$$\delta\phi = 0, \quad (C7)$$

$$\gamma_{ij} = a^2 e^{2\zeta} \delta_{ij}, \quad (C8)$$

one can use the Hamiltonian and momentum constraints to determine the scalar contributions to the lapse function and shift vector,

$$N = 1 + \alpha, \quad N_i = \partial_i \sigma, \quad (C9)$$

up to leading order. Substituting Eq. (C9) into the metric [Eq. (C1)] and expanding the action [Eq. (C2)] up to third order, we obtain the following,

$$\begin{aligned} S_3 = \int d^4x \, a^3 \left[a_1 \alpha^3 + a_2 \zeta \alpha^2 + a_3 \dot{\zeta} \alpha^2 + a_4 \frac{\partial^2 \sigma}{a^2} \alpha^2 + a_5 \frac{\partial \zeta \partial \sigma}{a^2} \alpha + a_6 \alpha \dot{\zeta} \zeta + a_7 \alpha \zeta \frac{\partial^2 \sigma}{a^2} + 3M_p^2 \alpha \dot{\zeta}^2 \right. \\ - \frac{M_p^2}{2} \frac{(\partial_i \partial_j \sigma)^2 - (\partial^2 \sigma)^2}{a^4} \alpha + 2M_p^2 H \zeta \frac{\partial \zeta \partial \sigma}{a^2} - 2M_p^2 \dot{\zeta} \alpha \frac{\partial^2 \sigma}{a^2} - 2M_p^2 \zeta \alpha \frac{\partial^2 \zeta}{a^2} - M_p^2 \zeta^2 \frac{\partial^2 \zeta}{a^2} - M_p^2 \alpha \frac{(\partial \zeta)^2}{a^2} \\ - M_p^2 \zeta \frac{(\partial \zeta)^2}{a^2} - 9M_p^2 \dot{\zeta}^2 \zeta + 2M_p^2 \dot{\zeta} \frac{\partial \zeta \partial \sigma}{a^2} + M_p^2 H \zeta^2 \frac{\partial^2 \sigma}{a^2} + 2M_p^2 \dot{\zeta} \zeta \frac{\partial^2 \sigma}{a^2} - \frac{M_p^2}{2} \frac{(\partial_i \partial_j \sigma)^2 - (\partial^2 \sigma)^2}{a^4} \zeta \\ \left. - 2M_p^2 \frac{\partial_i \zeta \partial_j \sigma \partial_i \partial_j \sigma}{a^4} \right], \quad (C10) \end{aligned}$$

where we defined the following coefficients,

$$\begin{aligned} a_1 &\equiv 3M_p^2 H^2 - \dot{\phi}^2 \left(\frac{1}{2} K_{,X} + \dot{\phi}^2 K_{,XX} + \frac{1}{6} \dot{\phi}^4 K_{,XXX} \right) - 2H \dot{\phi}^3 \left(5G_{,X} + \frac{11}{4} \dot{\phi}^2 G_{,XX} + \frac{1}{4} \dot{\phi}^4 G_{,XXX} \right) \\ &\quad + \dot{\phi}^2 \left(G_{,\phi} + \frac{7}{6} \dot{\phi}^2 G_{,X\phi} + \frac{1}{6} \dot{\phi}^4 G_{,\phi XX} \right), \\ a_2 &\equiv -9M_p^2 H^2 + 3\dot{\phi}^2 \left(\frac{1}{2} K_{,X} + \frac{1}{2} \dot{\phi}^2 K_{,XX} \right) + 18H \dot{\phi}^3 \left(G_{,X} + \frac{1}{4} \dot{\phi}^2 G_{,XX} \right) - 3\dot{\phi}^2 \left(G_{,\phi} + \frac{1}{2} \dot{\phi}^2 G_{,\phi X} \right), \\ a_3 &\equiv -6M_p^2 H + 6\dot{\phi}^3 \left(G_{,X} + \frac{1}{4} \dot{\phi}^2 G_{,XX} \right), \\ a_4 &\equiv 2M_p^2 H - 2\dot{\phi}^3 \left(G_{,X} + \frac{1}{4} \dot{\phi}^2 G_{,XX} \right), \\ a_5 &\equiv -2M_p^2 H + 3\dot{\phi}^3 G_{,X}, \\ a_6 &\equiv -9 \left(-2M_p^2 H + \dot{\phi}^3 G_{,X} \right), \\ a_7 &\equiv -2M_p^2 H + 3\dot{\phi}^3 G_{,X}. \end{aligned}$$

We note that the Hamiltonian and momentum constraints yield (these can also be obtained by varying the action above with respect to α and σ)

$$\alpha = \frac{2M_p^2 \dot{\zeta}}{u}, \quad (C11)$$

$$\partial^2 \sigma = a_8 \partial^2 \zeta + \partial^2 \chi, \quad (C12)$$

respectively, where we defined

$$u \equiv 2M_p^2 H - \dot{\phi}^3 G_{,X}, \quad (C13)$$

$$a_8 \equiv -\frac{2M_p^2}{u}, \quad (C14)$$

and where

$$\partial^2 \chi \equiv \frac{z^2 \dot{\zeta}}{M_p^2} = 3a^2 \dot{\zeta} + \frac{2M_p^2 a^2 \dot{\zeta}}{u^2} \left(-6M_p^2 H^2 + \dot{\phi}^2 K_{,X} + \dot{\phi}^4 K_{,XX} + 12H \dot{\phi}^3 G_{,X} + 3H \dot{\phi}^5 G_{,XX} - 2\dot{\phi}^2 G_{,\phi} - \dot{\phi}^4 G_{,X\phi} \right). \quad (C15)$$

If we substitute Eqs. (C11) and (C12) into the third order perturbed action [Eq. (C10)], we obtain

$$\begin{aligned} S_3 = \int d^4x \, a^3 & \left[A_1 \dot{\zeta}^3 + A_2 \dot{\zeta} \dot{\zeta}^2 + A_3 (\partial^2 \zeta) \dot{\zeta}^2 + A_4 \dot{\zeta} (\partial \zeta)^2 + A_5 \partial \zeta \partial \chi \dot{\zeta} + A_6 \dot{\zeta} \partial^2 \zeta + A_7 \dot{\zeta} (\partial \zeta)^2 + A_8 \partial \zeta \partial \chi \dot{\zeta} \right. \\ & + A_9 \dot{\zeta}^2 \partial^2 \chi + A_{10} \dot{\zeta}^2 \partial^2 \zeta + A_{11} \dot{\zeta}^2 \partial^2 \chi + A_{12} \partial_i \dot{\zeta} \partial_j \zeta \partial_i \partial_j \zeta + A_{13} \partial_i \dot{\zeta} \partial_j \zeta \partial_i \partial_j \chi + A_{14} \partial_i \dot{\zeta} \partial_j \chi \partial_i \partial_j \zeta \\ & + A_{15} \partial_i \dot{\zeta} \partial_j \chi \partial_i \partial_j \chi + (A_{16} \dot{\zeta} + A_{17} \zeta) (\partial_i \partial_j \zeta)^2 + (A_{18} \dot{\zeta} + A_{19} \zeta) (\partial_i \partial_j \chi)^2 + (A_{20} \dot{\zeta} + A_{21} \zeta) \partial_i \partial_j \zeta \partial_i \partial_j \chi \\ & \left. + (A_{22} \dot{\zeta} + A_{23} \zeta) (\partial^2 \zeta)^2 + (A_{24} \dot{\zeta} + A_{25} \zeta) (\partial^2 \chi)^2 + (A_{26} \dot{\zeta} + A_{27} \zeta) \partial^2 \zeta \partial^2 \chi + A_{28} \dot{\zeta} \partial^2 \chi \right] + S_b, \quad (C16) \end{aligned}$$

where we defined the following,

$$\begin{aligned} A_1 &\equiv \frac{(2M_p^2)^3}{u^3} a_1 + \frac{(2M_p^2)^2}{u^2} a_3 + 6 \frac{M_p^4}{u}, \quad A_2 \equiv \frac{(2M_p^2)^2}{u^2} a_2 + \frac{2M_p^2}{u} a_6 - 9M_p^2, \\ A_3 &\equiv \frac{(2M_p^2)^2}{u^2 a^2} a_4 a_8 - \frac{(2M_p^2)^2}{u a^2} a_8, \quad A_4 \equiv \frac{2M_p^2}{u a^2} a_5 a_8 - \frac{2M_p^4}{u a^2} + \frac{2M_p^2}{a^2} a_8, \\ A_5 &\equiv \frac{2M_p^2}{u a^2} a_5 + \frac{2M_p^2}{a^2}, \quad A_6 \equiv \frac{2M_p^2}{u a^2} a_8 a_7 - \frac{(2M_p^2)^2}{u a^2} + \frac{2M_p^2}{a^2} a_8, \\ A_7 &\equiv \frac{2M_p^2}{a^2} H a_8 - \frac{M_p^2}{a^2}, \quad A_8 \equiv \frac{2M_p^2 H}{a^2}, \quad A_9 \equiv \frac{(2M_p^2)^2}{u^2 a^2} a_4 - \frac{(2M_p^2)^2}{u a^2}, \\ A_{10} &\equiv -\frac{M_p^2}{a^2} + \frac{M_p^2}{a^2} H a_8, \quad A_{11} \equiv \frac{M_p^2 H}{a^2}, \quad \frac{A_{12}}{a_8} \equiv A_{13} \equiv A_{14} \equiv -\frac{2M_p^2}{a^4} a_8, \\ A_{15} &\equiv -\frac{2M_p^2}{a^4}, \quad A_{16} \equiv -\frac{M_p^4}{u a^4} a_8^2, \quad A_{17} \equiv -\frac{M_p^2}{2a^4} a_8^2, \quad A_{18} \equiv -\frac{M_p^4}{u a^4}, \\ A_{19} &\equiv -\frac{M_p^2}{2a^4}, \quad A_{20} \equiv -\frac{2M_p^4}{u a^4} a_8, \quad A_{21} \equiv -\frac{M_p^2}{a^4} a_8, \quad A_{22} \equiv \frac{M_p^4}{u a^4} a_8^2, \\ A_{23} &\equiv \frac{M_p^2}{2a^4} a_8^2, \quad A_{24} \equiv \frac{M_p^4}{u a^4}, \quad A_{25} \equiv \frac{M_p^2}{2a^4}, \quad A_{26} \equiv \frac{2M_p^4}{u a^4} a_8, \\ A_{27} &\equiv \frac{M_p^2}{a^4} a_8, \quad A_{28} \equiv \frac{2M_p^2}{a^2} + \frac{2M_p^2}{u a}. \end{aligned}$$

We note that S_b is a short-hand notation for all the boundary terms, which do not make a contribution to the calculation at 3rd order. After many integrations by part, we obtain

$$\begin{aligned} S_3 = \int d^4x \, & \left(B_1 [\partial \zeta \partial \chi \partial^2 \zeta - \zeta \partial_i \partial_j (\partial_i \zeta \partial_j \chi)] + B_2 \dot{\zeta}^2 \partial^2 \zeta \right. \\ & \left. + B_3 \dot{\zeta} \partial \zeta \partial \chi + B_4 \zeta (\partial_i \partial_j \chi)^2 + B_5 \zeta (\partial \zeta)^2 + B_6 \dot{\zeta}^3 + B_7 \dot{\zeta}^2 \zeta - 2f(\zeta) \frac{\delta \mathcal{L}_2}{\delta \zeta} \right), \quad (C17) \end{aligned}$$

where

$$\frac{\delta \mathcal{L}_2}{\delta \zeta} = \partial_t(az^2\dot{\zeta}) - \frac{c_s^2 z^2}{a} \partial^2 \zeta, \quad (\text{C18})$$

and where

$$f(\zeta) = \frac{A_{20}a^2}{4M_p^2} [(\partial\zeta)^2 - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\zeta)] + \frac{A_{18}a^2}{M_p^2} [\partial\zeta\partial\chi - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\chi)] - \frac{2A_4a^3 - C_1}{2z^2c_s^2} a\zeta\dot{\zeta}. \quad (\text{C19})$$

We also introduced the following,

$$\begin{aligned} B_1 &\equiv -A_{21}a^3 - \frac{1}{2}\partial_t(A_{20}a^3) + \frac{A_{20}}{2}a^3H - 2A_{18}z^2c_s^2a, \\ B_2 &\equiv A_3a^3 + (A_{26} + A_{20})\frac{z^2a^3}{M_p^2}, \\ B_3 &\equiv A_5a^3 + A_{15}\frac{z^2a^3}{M_p^2}, \\ B_4 &\equiv -\partial_t(A_{18}a^3) - 3A_{19}a^3 + 2A_{18}Ha^3, \\ B_5 &\equiv \partial_t\left(A_4 + A_{13}\frac{z^2}{2M_p^2}\right)a^3 - 2A_{10}a^3 + A_7a^3, \\ B_6 &\equiv A_1a^3 + (A_{18} + A_{24})\left(\frac{z^2}{M_p^2}\right)^2a^3 + A_9\frac{z^2a^3}{M_p^2} + \frac{2A_4a^3 - C_1}{2c_s^2}a^2, \\ B_7 &\equiv A_2a^3 + [A_{15}a^3 + \partial_t(A_{18}a)a^2 - \partial_t(A_{18}a^3) - 3A_{19}a^3 + 2A_{18}Ha^3 + A_{25}a^3]\left(\frac{z^2}{M_p^2}\right)^2 \\ &\quad + \frac{az^2}{2}\partial_t\left[\frac{(2A_4a^3 - C_1)a}{c_s^2z^2}\right] - \partial_t\left(\frac{az^2}{2}\right)\frac{(2A_4a^3 - C_1)a}{c_s^2z^2} - B_4\left(\frac{z^2}{M_p^2}\right)^2. \end{aligned} \quad (\text{C20})$$

Furthermore,

$$C_1 \equiv A_6a^3 + (A_{27} - A_{21})\frac{z^2a^3}{M_p^2}, \quad (\text{C21})$$

and c_s is the speed of sound introduced earlier in Eq. (59).

2. Third order perturbed action in the limit of the matter-dominated contracting phase

Let us evaluate the third order action given by Eq. (C17) in a matter-dominated contracting phase when $G(\phi, X) = 0$ and $K(\phi, X) = M_p^2X - V(\phi)$. In this case, we have

$$\begin{aligned} a_1 &= 3M_p^2H^2 - \frac{1}{2}M_p^2\dot{\phi}^2, \quad a_2 = -9M_p^2H^2 + \frac{3}{2}M_p^2\dot{\phi}^2, \quad a_3 = -6M_p^2H, \\ a_4 &= 2M_p^2H, \quad a_5 = -2M_p^2H, \quad a_6 = 18M_p^2H, \quad a_7 = -2M_p^2H, \end{aligned}$$

together with $u = 2M_p^2H$, $a_8 = -2M_p^2/u$, and $\partial^2\chi = z^2\dot{\zeta}/M_p^2 = a^2\dot{\phi}^2\dot{\zeta}/2H^2$. Then,

$$\begin{aligned} A_1 &= -\frac{M_p^2\dot{\phi}^2}{2H^3}, \quad A_2 = \frac{3M_p^2\dot{\phi}^2}{2H^2}, \quad A_3 = 0, \quad A_4 = -\frac{M_p^2}{a^2H}, \quad A_5 = 0, \quad A_6 = \frac{2M_p^2}{a^2H}, \\ A_7 &= -\frac{3M_p^2}{a^2}, \quad A_8 = \frac{2M_p^2H}{a^2}, \quad A_9 = 0, \quad A_{10} = -\frac{2M_p^2}{a^2}, \quad A_{11} = \frac{M_p^2H}{a^2}, \quad A_{12} = A_{13} = A_{14} = \frac{2M_p^2}{a^4H}, \\ A_{15} &= -\frac{2M_p^2}{a^4}, \quad A_{16} = -\frac{M_p^4}{ha^4}a_8^2, \quad A_{17} = -\frac{M_p^2}{2a^4}a_8^2, \quad A_{18} = -\frac{M_p^2}{2a^4H}, \quad A_{19} = -\frac{M_p^2}{2a^4}, \\ A_{20} &= -\frac{2M_p^4}{ha^4}a_8, \quad A_{21} = -\frac{M_p^2}{a^4}a_8, \quad A_{22} = \frac{M_p^4}{ha^4}a_8^2, \quad A_{23} = \frac{M_p^2}{2a^4}a_8^2, \quad A_{24} = \frac{M_p^4}{ha^4}, \quad A_{25} = \frac{M_p^2}{2a^4}, \\ A_{26} &= \frac{2M_p^4}{ha^4}a_8, \quad A_{27} = \frac{M_p^2}{a^4}a_8, \quad A_{28} = \frac{2M_p^2}{a^2} + \frac{2M_p^2}{ha}a_7. \end{aligned}$$

Thus,

$$\begin{aligned}
B_1(t) &= M_p^2 \frac{\dot{\phi}(t)^2 + 2\dot{H}(t)}{2a(t)H(t)^3} = 0, \\
B_2(t) &= 0, \\
B_3(t) &= -\frac{M_p^2 a(t) \dot{\phi}(t)^2}{H(t)^2} = -2\epsilon(t) M_p^2 a(t), \\
B_4(t) &= -\frac{M_p^2 \dot{H}(t)}{2a(t)H(t)^2} = \frac{M_p^2 \epsilon(t)}{2a(t)}, \\
B_5(t) &= M_p^2 a(t) \frac{2H(t)^2 \dot{H}(t) + 2\dot{\phi}(t)H(t)\ddot{\phi}(t) - \dot{\phi}(t)^2 [3\dot{H}(t) - H(t)^2]}{2H(t)^4} = M_p^2 \epsilon(t)^2 a(t), \\
B_6(t) &= 0, \\
B_7(t) &= \frac{M_p^2 a(t)^3 \dot{\phi}(t) \left\{ \dot{\phi}(t) [\dot{\phi}(t)^2 + 4H(t)^2] \dot{H}(t) - 8H(t)^3 \ddot{\phi}(t) \right\}}{8H(t)^6} = -\frac{1}{2} M_p^2 [\epsilon^3(t) - 2\epsilon^2(t)] a(t)^3.
\end{aligned}$$

Here, we consider $\dot{\phi}(t)^2 = 2\epsilon(t)H(t)^2$, $\epsilon(t) = -\dot{H}(t)/H(t)^2$, and $c_s = 1$. Therefore, we find that

$$S_3 = \int d^4x \left[-2\epsilon M_p^2 a(t) \partial \zeta \dot{\zeta} \partial \chi + \frac{M_p^2 \epsilon}{2a(t)} \zeta (\partial_i \partial_j \chi)^2 + M_p^2 \epsilon^2 a(t) \zeta (\partial \zeta)^2 - \frac{1}{2} M_p^2 (\epsilon^3 - 2\epsilon^2) a(t)^3 \dot{\zeta}^2 \zeta - 2f(\zeta) \frac{\delta \mathcal{L}}{\delta \zeta} \right], \quad (\text{C22})$$

and

$$f(\zeta) = \frac{1}{4a(t)^2 H(t)^2} [(\partial \zeta)^2 - \partial^{-2} \partial_i \partial_j (\partial_i \zeta \partial_j \zeta)] + \frac{1}{2a(t)^2 H(t)} [-\partial \zeta \partial \chi + \partial^{-2} \partial_i \partial_j (\partial_i \zeta \partial_j \chi)] - \frac{1}{H(t)} \zeta \dot{\zeta}. \quad (\text{C23})$$

This is equivalent to the third order action given in [30] noting that we defined $\partial^2 \chi = a^2 \epsilon \dot{\zeta}$ whereas they considered $\partial^2 \chi = \dot{\zeta}$.

Appendix D: EVALUATING THE SHAPE FUNCTION IN THE BOUNCE PHASE

We want to evaluate Eq. (95) which, as explained in the text, has three dominant terms: the $\zeta \zeta'^2$ term, the ζ'^3 term, and the field redefinition term. Let us start with the contribution from the $\zeta \zeta'^2$ term to the shape function, which is

$$\begin{aligned}
\mathcal{A}_{\zeta \zeta'^2} &= \frac{\zeta_{k_3}^*(\eta_+)}{4\zeta_{k_1}(\eta_+) \zeta_{k_2}(\eta_+)} \\
&\times i \int_{\eta_-}^{\eta_+} d\eta \left[\left(\frac{B_3(\eta) z^2(\eta)}{M_p^2 a_B} k_3^3 \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta) z^4(\eta)}{M_p^4 a_B} k_3^3 \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + \frac{B_7(\eta)}{a_B} k_3^3 \right) \zeta_{k_1}(\eta) \zeta'_{k_2}(\eta) \zeta'_{k_3}(\eta) \right], \quad (\text{D1})
\end{aligned}$$

where we omit the 5 additional permutations for now. Using Eq. (89) for $\zeta_{k_i}(\eta)$, we get

$$\begin{aligned}
\mathcal{A}_{\zeta \zeta'^2} &= \frac{ik_3^3}{4a_B} \frac{\zeta_{k_3}^*(\eta_+) \zeta_{k_2}^{m'}(\eta_{B-}) \zeta_{k_3}^{m'}(\eta_{B-})}{\zeta_{k_1}(\eta_+) \zeta_{k_2}(\eta_+)} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^4 \\
&\times \int_{\eta_-}^{\eta_+} d\eta \left[\left(\frac{B_3(\eta) z^2(\eta)}{M_p^2} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta) z^4(\eta)}{M_p^4} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + B_7(\eta) \right) \zeta_{k_1}(\eta) \right]. \quad (\text{D2})
\end{aligned}$$

We note that taking ζ'_{k_i} from Eq. (89) actually gives an upper bound on $\mathcal{A}_{\zeta \zeta'^2}$ since Eq. (89) used the maximal growth rate (79) for the full range $[\eta_-, \eta_+]$. This introduces some small error in the final result but this will turn out to be unimportant since, as we will see, the field redefinition contribution to the shape function will dominate over this upper bound on $\mathcal{A}_{\zeta \zeta'^2}$.

The time-dependent terms that remain inside the integral are B_3 , B_4 , B_7 , z^2 , and ζ_{k_1} . The latter, ζ_{k_1} , may experience a nontrivial enhancement during the interval $[\eta_-, \eta_+]$, and consequently, it may contribute significantly to

the integral. The other terms, i.e. $B_3 z^2$, $B_4 z^4$, and B_7 defined in Eqs. (C20) and (71), contribute as almost constant terms in the integral over the range $[\eta_-, \eta_+]$. Therefore, we rewrite Eq. (D2) in the following form,

$$\mathcal{A}_{\zeta\zeta'^2} \simeq \frac{ik_3^3}{4a_B} \frac{\zeta_{k_3}^*(\eta_+) \zeta_{k_2}^{\text{m}'}(\eta_{B-}) \zeta_{k_3}^{\text{m}'}(\eta_{B-})}{\zeta_{k_1}(\eta_+) \zeta_{k_2}(\eta_+)} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^4 \left(\frac{B_3(\eta_B) z_B^2}{M_p^2} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta_B) z_B^4}{M_p^4} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + B_7(\eta_B) \right) \times \int_{\eta_-}^{\eta_+} d\eta \left[\zeta_{k_1}^{\text{m}}(\eta_{B-}) + \zeta_{k_1}^{\text{m}'}(\eta_{B-}) \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 (\eta - \eta_-) \right], \quad (\text{D3})$$

where we denote $z_B \equiv z(\eta_B)$. Performing the integral and using Eq. (89) for $\zeta_{k_i}(\eta_+)$ (again, this contributes to obtaining an upper bound for $\mathcal{A}_{\zeta\zeta'^2}$), we obtain

$$\mathcal{A}_{\zeta\zeta'^2} \simeq \frac{ik_3^3}{2a_B} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^4 \left(\frac{B_3(\eta_B) z_B^2}{M_p^2} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta_B) z_B^4}{M_p^4} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + B_7(\eta_B) \right) \frac{\zeta_{k_2}^{\text{m}'} \zeta_{k_3}^{\text{m}'} \zeta_{k_3}^{\text{m}*}}{\zeta_{k_2}^{\text{m}}} \Delta\eta_{\text{amp}} \times \left[1 + \frac{\zeta_{k_1}^{\text{m}'}}{\zeta_{k_1}^{\text{m}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \Delta\eta_{\text{amp}} \right] \left[1 + 2 \frac{\zeta_{k_3}^{\text{m}*}}{\zeta_{k_3}^{\text{m}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \Delta\eta_{\text{amp}} \right] \left[1 + 2 \frac{\zeta_{k_1}^{\text{m}'}}{\zeta_{k_1}^{\text{m}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \Delta\eta_{\text{amp}} \right]^{-1} \times \left[1 + 2 \frac{\zeta_{k_2}^{\text{m}'}}{\zeta_{k_2}^{\text{m}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \Delta\eta_{\text{amp}} \right]^{-1}, \quad (\text{D4})$$

where the modes $\zeta_{k_i}^{\text{m}}$ are implicitly evaluated at η_{B-} and where we recall that $2\Delta\eta_{\text{amp}} = \eta_+ - \eta_-$. At this point, one could substitute $\zeta_{k_i}^{\text{m}}(\eta_{B-})$ with Eq. (90) and write the full expression for $\mathcal{A}_{\zeta\zeta'^2}$. However, to satisfy the observational bound on the tensor-to-scalar ratio, we expect there to be a large amplification of curvature perturbations during the interval $[\eta_-, \eta_+]$. In fact, from Eqs. (31) and (81), it must be that $|\zeta_{k_i}^{\text{m}'} / \zeta_{k_i}^{\text{m}}| [\phi'_B / \phi'(\eta_-)]^2 \Delta\eta_{\text{amp}} \gg \mathcal{O}(1)$. In that limit, the shape function (D4) reduces to (to leading order)

$$\mathcal{A}_{\zeta\zeta'^2} \simeq \frac{ik_3^3}{4a_B} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^4 \left(\frac{B_3(\eta_B) z_B^2}{M_p^2} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta_B) z_B^4}{M_p^4} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + B_7(\eta_B) \right) \Delta\eta_{\text{amp}} |\zeta_{k_3}^{\text{m}'}|^2, \quad (\text{D5})$$

which is purely imaginary, and hence, does not physically contribute to the physical shape function. The next-to-leading order terms are

$$\mathcal{A}_{\zeta\zeta'^2} \simeq \frac{ik_3^3}{8a_B} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \left(\frac{B_3(\eta_B) z_B^2}{M_p^2} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta_B) z_B^4}{M_p^4} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + B_7(\eta_B) \right) \times \frac{\zeta_{k_2}^{\text{m}'} \zeta_{k_3}^{\text{m}'} \zeta_{k_3}^{\text{m}*}}{\zeta_{k_2}^{\text{m}}} \left[\frac{\zeta_{k_1}^{\text{m}'} + 2 \frac{\zeta_{k_3}^{\text{m}*}}{\zeta_{k_3}^{\text{m}}} - \frac{\zeta_{k_1}^{\text{m}'} \zeta_{k_3}^{\text{m}*}}{\zeta_{k_1}^{\text{m}} \zeta_{k_3}^{\text{m}}} \left(\frac{\zeta_{k_1}^{\text{m}2} \zeta_{k_2}^{\text{m}}}{\zeta_{k_1}^{\text{m}2} \zeta_{k_2}^{\text{m}}} + \frac{\zeta_{k_1}^{\text{m}} \zeta_{k_2}^{\text{m}2}}{\zeta_{k_1}^{\text{m}2} \zeta_{k_2}^{\text{m}}} \right) \right]. \quad (\text{D6})$$

Using Eq. (90) for $\zeta_{k_i}^{\text{m}}(\eta_{B-})$ and taking the limit $k \ll \mathcal{H}$, we find the leading order real-valued contribution to be

$$\mathcal{A}_{\zeta\zeta'^2} \simeq -\frac{A^2}{16a_B \Delta\eta_{B-}^4} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 (-k_1^3 + k_2^3 + k_3^3) \left(\frac{B_3(\eta_B) z_B^2}{M_p^2} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_3^2} + \frac{B_4(\eta_B) z_B^4}{M_p^4} \frac{(\vec{k}_2 \cdot \vec{k}_3)^2}{k_3^2 k_2^2} + B_7(\eta_B) \right). \quad (\text{D7})$$

The $B_n(\eta_B)$ terms can be evaluated using Eqs. (C20), (71), and (55):

$$\frac{B_3(\eta_B) z_B^2}{M_p^2} \simeq -\frac{6\beta M_p^4 a_B^5 (3\beta M_p^2 a_B^2 + 2\gamma^2 \phi_B'^2)}{\gamma^4 \phi_B'^4}, \quad (\text{D8})$$

$$\frac{B_4(\eta_B) z_B^4}{M_p^4} \simeq -\frac{9\beta^2 M_p^6 a_B^7 (4M_p^4 \Upsilon a_B^6 - 3\gamma^2 \phi_B'^6)}{2\gamma^6 \phi_B'^{10}}, \quad (\text{D9})$$

$$B_7(\eta_B) \simeq \frac{3M_p^2 a_B^3}{2\gamma^6 \phi_B'^{10}} (-9a_B^4 \beta^2 \gamma^2 M_p^4 \phi_B'^6 + 6a_B^2 \beta \gamma^4 M_p^2 \phi_B'^8 + 6\gamma^6 \phi_B'^{10} + 12a_B^{10} \beta^2 M_p^8 \Upsilon + 24a_B^8 \beta \gamma^2 M_p^6 \phi_B'^2 \Upsilon + 32a_B^6 \gamma^4 M_p^4 \phi_B'^4 \Upsilon). \quad (\text{D10})$$

Similarly, we can use the previous procedure to find the contribution from the ζ'^3 term to the shape function (again, omitting the additional perturbations for now),

$$\begin{aligned}
\mathcal{A}_{\zeta'^3} &= \frac{ik_3^3 \zeta_{k_3}^*(\eta_+)}{4\zeta_{k_1}(\eta_+)\zeta_{k_2}(\eta_+)} \int_{\eta_-}^{\eta_+} d\eta \left(\frac{B_6(\eta)}{a_B^2} \zeta'_{k_1}(\eta) \zeta'_{k_2}(\eta) \zeta'_{k_3}(\eta) \right) \\
&= \frac{ik_3^3 \zeta_{k_3}^*(\eta_+)}{2\zeta_{k_1}(\eta_+)\zeta_{k_2}(\eta_+)} \frac{B_6(\eta_B)}{a_B^2} \zeta_{k_1}^{\text{m}'} \zeta_{k_2}^{\text{m}'} \zeta_{k_3}^{\text{m}'} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^6 \Delta\eta_{\text{amp}} \\
&= \frac{ik_3^3 B_6(\eta_B)}{2a_B^2} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^4 \left\{ |\zeta_{k_3}^{\text{m}'}|^2 + \frac{\zeta_{k_3}^{\text{m}'} \zeta_{k_3}^{\text{m}*}}{2\Delta\eta_{\text{amp}}} \left(\frac{\phi'(\eta_-)}{\phi'_B} \right)^2 \left[1 - \frac{\zeta_{k_3}^{\text{m}*}}{\zeta_{k_3}^{\text{m}}} \left(\frac{\zeta_{k_1}^{\text{m}}}{\zeta_{k_1}^{\text{m}'} } + \frac{\zeta_{k_2}^{\text{m}}}{\zeta_{k_2}^{\text{m}'} } \right) \right] + \dots \right\} \\
&\simeq - \frac{A^2 B_6(\eta_B)}{16a_B^2 \Delta\eta_{\text{amp}} \Delta\eta_{B-}^4} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 (k_1^3 + k_2^3 + k_3^3) .
\end{aligned} \tag{D11}$$

The ellipsis in the third line denotes higher-order terms in $|\zeta_{k_i}^{\text{m}}/\zeta_{k_i}^{\text{m}'}|[\phi'(\eta_-)/\phi'_B]^2(\Delta\eta_{\text{amp}})^{-1}$, and in the fourth line, we took the leading order real-valued term in the expansion. From Eq. (C20), the B_6 term is given by

$$B_6(\eta_B) \simeq \frac{10M_p^4 a_B^6 [2\beta M_p^2 a_B^2 + 3\gamma^2 \phi_B'^2]}{\gamma^3 \phi_B'^5} . \tag{D12}$$

From the same argument as for $\mathcal{A}_{\zeta\zeta'^2}$, the result (D11) is actually an upper bound (in absolute value) for $\mathcal{A}_{\zeta'^3}$, which we will comment on later.

The contribution from the field redefinition term to the shape function is (again, omitting the additional perturbations for now)

$$\begin{aligned}
\mathcal{A}_{\text{redef}} &= k_3^3 \left[-\frac{A_{18}(\eta_+) a_B z^2(\eta_+)}{4M_p^4} \left(\frac{\vec{k}_1 \cdot (\vec{k}_3 - \vec{k}_1)}{k_1^2} - \frac{(\vec{k}_1 \cdot \vec{k}_3)[(\vec{k}_3 - \vec{k}_1) \cdot \vec{k}_3]}{k_1^2 k_3^2} \right) + \frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{8z^2(\eta_+) c_s^2} \right] \frac{\zeta'_{k_1}(\eta_+)}{\zeta_{k_1}(\eta_+)} \\
&= k_3^3 \left[-\frac{A_{18}(\eta_+) a_B z^2(\eta_+)}{4M_p^4} \left(\frac{\vec{k}_1 \cdot (\vec{k}_3 - \vec{k}_1)}{k_1^2} - \frac{(\vec{k}_1 \cdot \vec{k}_3)[(\vec{k}_3 - \vec{k}_1) \cdot \vec{k}_3]}{k_1^2 k_3^2} \right) + \frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{8z^2(\eta_+) c_s^2} \right] \\
&\quad \times \frac{\zeta_{k_1}^{\text{m}'}}{\zeta_{k_1}^{\text{m}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \left[1 + \frac{\zeta_{k_1}^{\text{m}'}}{\zeta_{k_1}^{\text{m}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 2\Delta\eta_{\text{amp}} \right]^{-1} \\
&\simeq \frac{k_3^3}{2\Delta\eta_{\text{amp}}} \left[-\frac{A_{18}(\eta_+) a_B z^2(\eta_+)}{4M_p^4} \left(\frac{\vec{k}_1 \cdot (\vec{k}_3 - \vec{k}_1)}{k_1^2} - \frac{(\vec{k}_1 \cdot \vec{k}_3)[(\vec{k}_3 - \vec{k}_1) \cdot \vec{k}_3]}{k_1^2 k_3^2} \right) + \frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{8z^2(\eta_+) c_s^2} \right] ,
\end{aligned} \tag{D13}$$

where we took the leading order term in $|\zeta_{k_1}^{\text{m}}/\zeta_{k_1}^{\text{m}'}|[\phi'(\eta_-)/\phi'_B]^2(\Delta\eta_{\text{amp}})^{-1}$. Here,

$$\frac{A_{18}(\eta_+) a_B z^2(\eta_+)}{M_p^4} \simeq \frac{3a_B^4 \beta M_p^4}{\gamma^3 \phi_B'^5(\eta_+)} , \tag{D14}$$

$$\frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{z^2(\eta_+) c_s^2} \simeq \frac{4}{\beta \gamma \phi_B'^3(\eta_+)} [3a_B^2 M_p^2 \beta + 2\gamma^2 \phi_B'^2(\eta_+)] . \tag{D15}$$

Recalling Eq. (54) and the fact that $|\eta_+ - \eta_B| = |\eta_- - \eta_B| = \Delta\eta_{\text{amp}}$, we have $\phi'(\eta_+) = \phi'(\eta_-)$, and so we can rewrite the terms above as

$$\frac{A_{18}(\eta_+) a_B z^2(\eta_+)}{M_p^4} \simeq \frac{3a_B^4 \beta M_p^4}{\gamma^3 \phi_B'^5} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^5 , \tag{D16}$$

$$\frac{2A_4(\eta_+) a_B^3 - C_1(\eta_+)}{z^2(\eta_+) c_s^2} \simeq \frac{12a_B^2 M_p^2}{\gamma \phi_B'^3} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^3 + \frac{8\gamma}{\beta \phi_B'} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right) . \tag{D17}$$

Combining the different contributions (including all permutations), the general form of the total shape function is

found to be

$$\begin{aligned}
\mathcal{A}(k_1, k_2, k_3) \simeq & \frac{-A^2}{16\Delta\eta_{B-}^4} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \frac{1}{\prod_i k_i^2} \left[\frac{B_3(\eta_B)z_B^2}{2M_p^2 a_B} \left(2 \sum_{i \neq j} k_i^7 k_j^2 - 2 \sum_{i \neq j} k_i^5 k_j^4 - \sum_{i \neq j \neq \ell} k_i^5 k_j^2 k_\ell^2 \right) \right. \\
& + \frac{B_4(\eta_B)z_B^4}{4M_p^4 a_B} \left(- \sum_i k_i^9 + 2 \sum_{i \neq j} k_i^6 k_j^3 + 6 \sum_{i \neq j} k_i^7 k_j^2 - 6 \sum_{i \neq j} k_i^5 k_j^4 - 2 \sum_{i \neq j \neq \ell} k_i^4 k_j^3 k_\ell^2 + 2 \sum_{i \neq j \neq \ell} k_i^5 k_j^2 k_\ell^2 \right) \Big] \\
& - \left[\frac{A^2}{8\Delta\eta_{B-}^4} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \frac{B_7(\eta_B)}{a_B} + \frac{3A^2}{8\Delta\eta_{B-}^4 \Delta\eta_{\text{amp}}} \left(\frac{\phi'_B}{\phi'(\eta_-)} \right)^2 \frac{B_6(\eta_B)}{a_B^2} - \frac{2A_4(\eta_+)a_B^3 - C_1(\eta_+)}{8z^2(\eta_+)c_s^2 \Delta\eta_{\text{amp}}} \right] \sum_i k_i^3 \\
& - \frac{A_{18}(\eta_+)a_B z^2(\eta_+)}{32M_p^4 \Delta\eta_{\text{amp}}} \frac{1}{\prod_i k_i^2} \left(\sum_{i \neq j} k_i^7 k_j^2 - 2 \sum_{i \neq j} k_i^5 k_j^4 - 2 \sum_{i \neq j \neq \ell} k_i^5 k_j^2 k_\ell^2 + \sum_{i \neq j} k_i^6 k_j^3 - \sum_{i \neq j \neq \ell} k_i^4 k_j^3 k_\ell^2 \right). \tag{D18}
\end{aligned}$$

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